Dynamics of Arithmetic
A Connectionist View of Arithmetic Skills

Richard Dallaway

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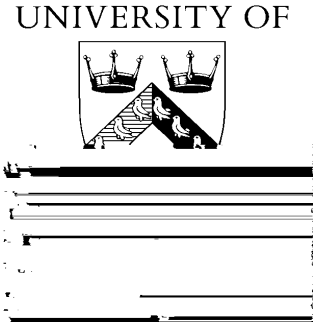

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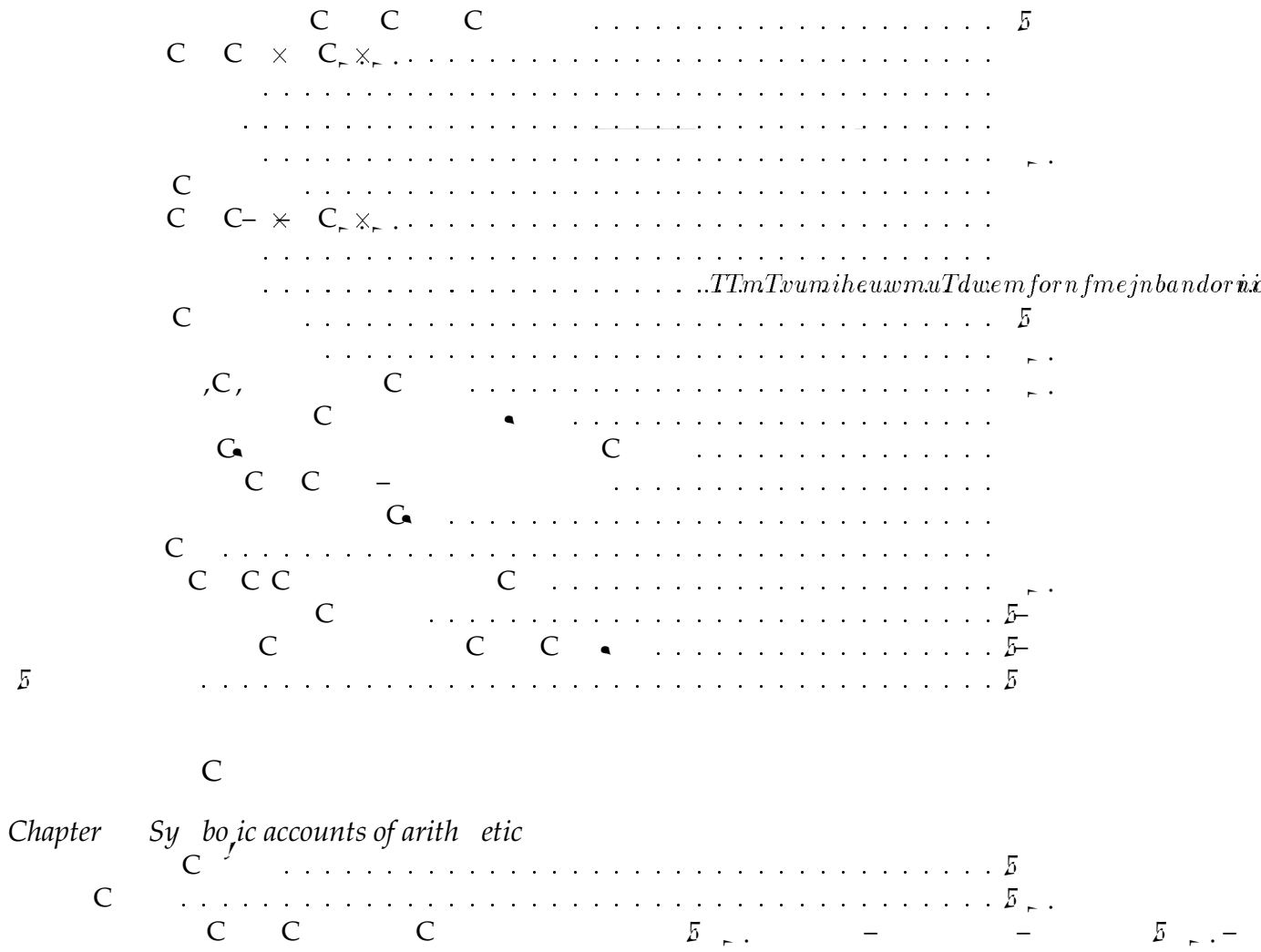
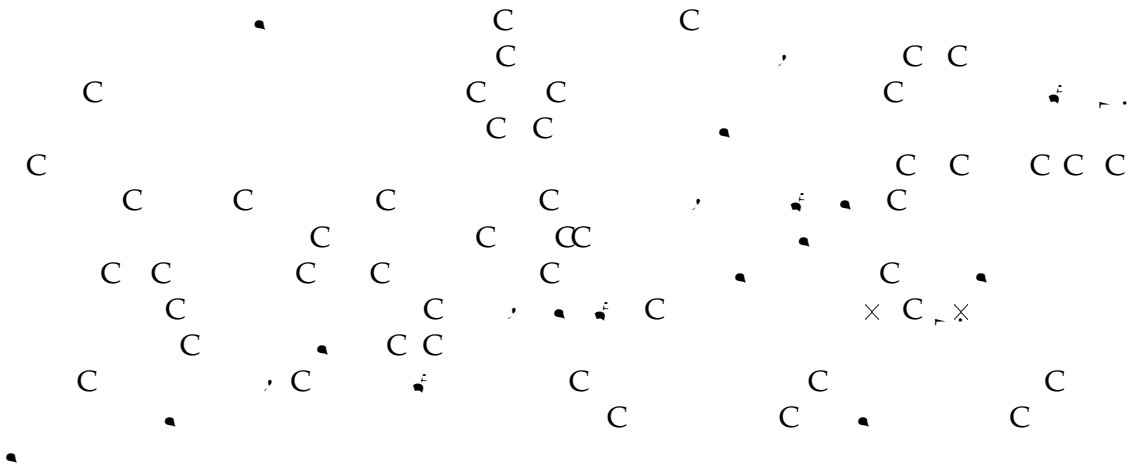


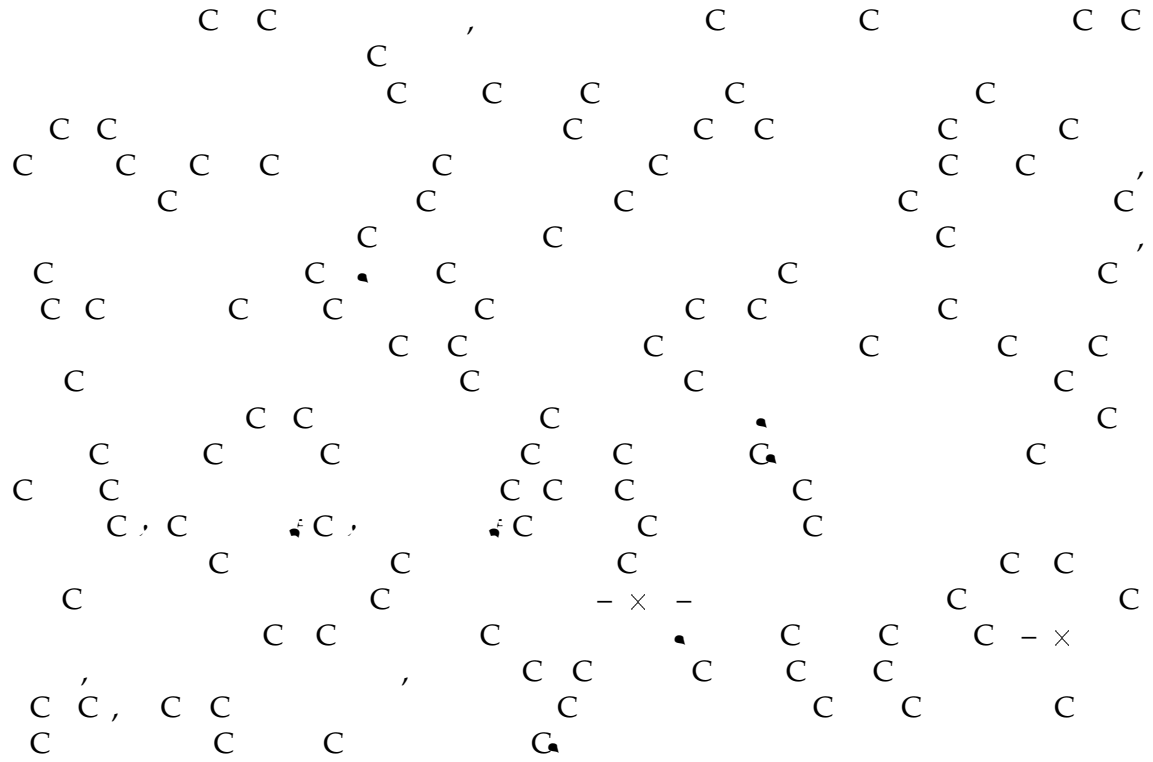
C

C

C C C .

Introduction





1.2 Part II—Multicolumn multiplication

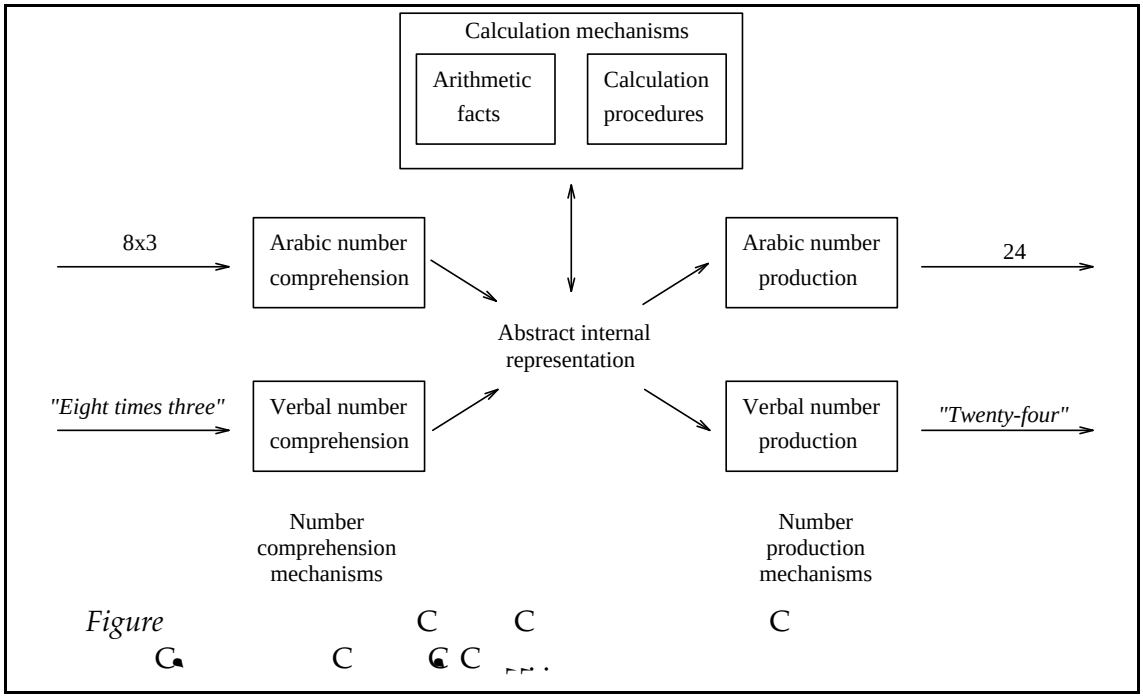


Figure C C C

C C C C . .

C C C C C C C C C ,

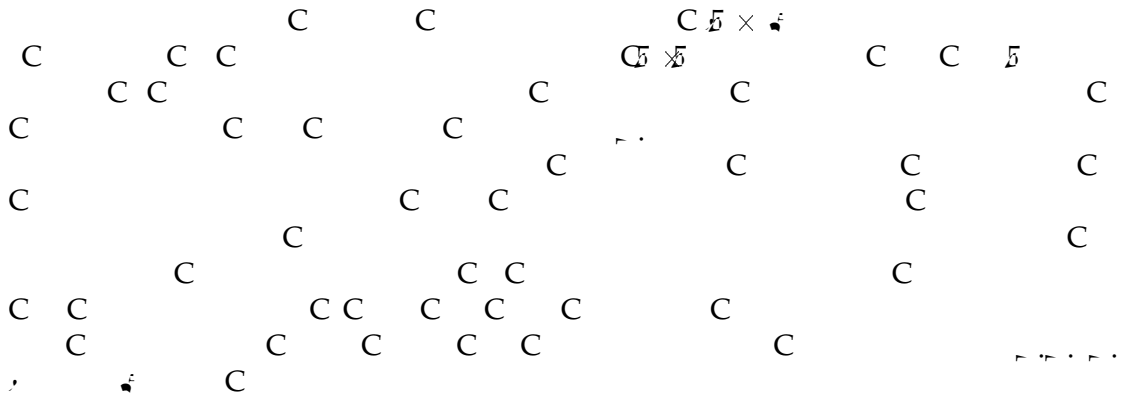
C C C C C C

C C C C C C C C

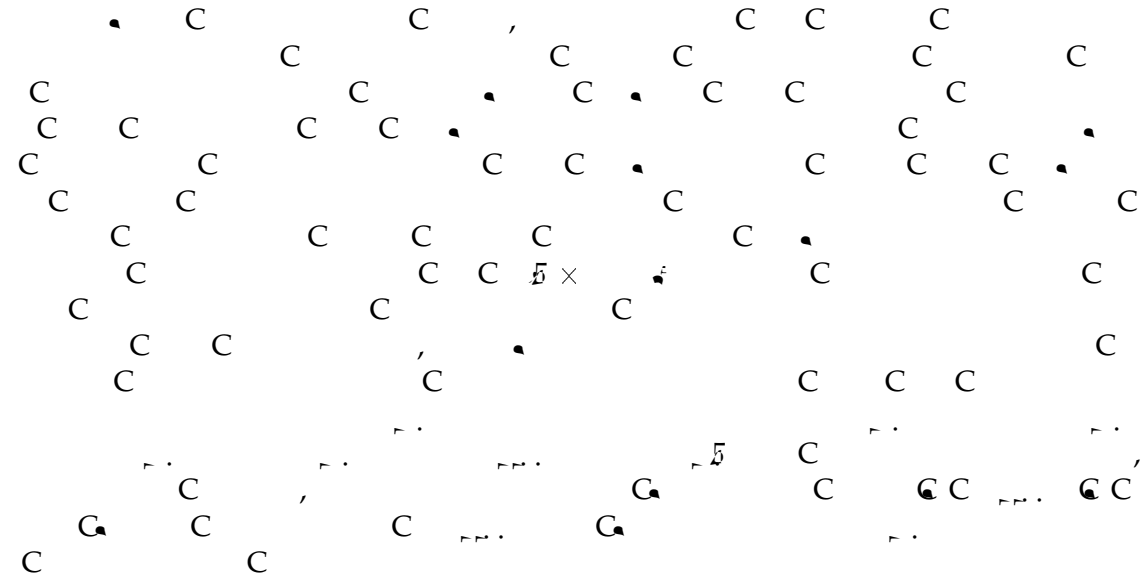
C C C C C C C C C

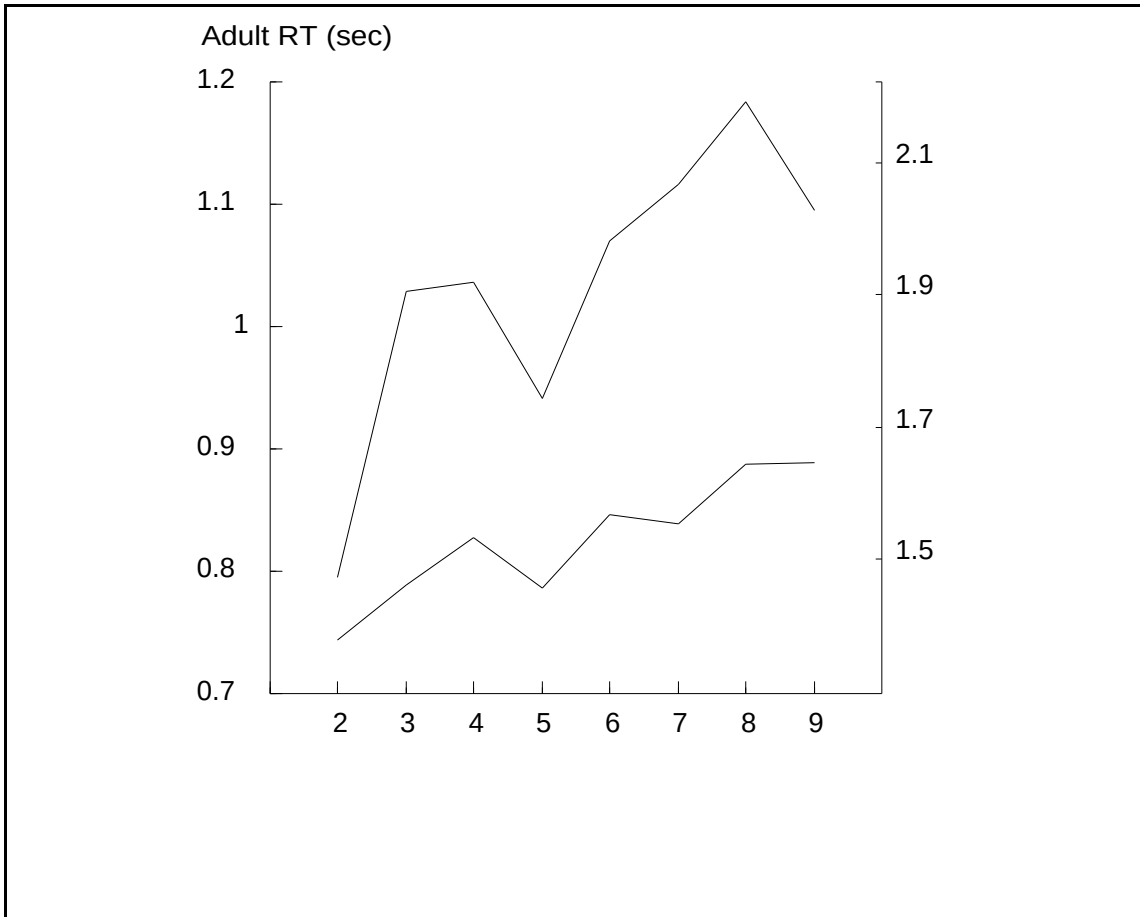
Menta, Arith etic

Me ory for Arith etic Facts



2.1 Phenomena




Adult RT (sec)
1.2
1.1
1
0.9
0.8
0.7
2.1
1.9
1.7
1.5
2
3
4
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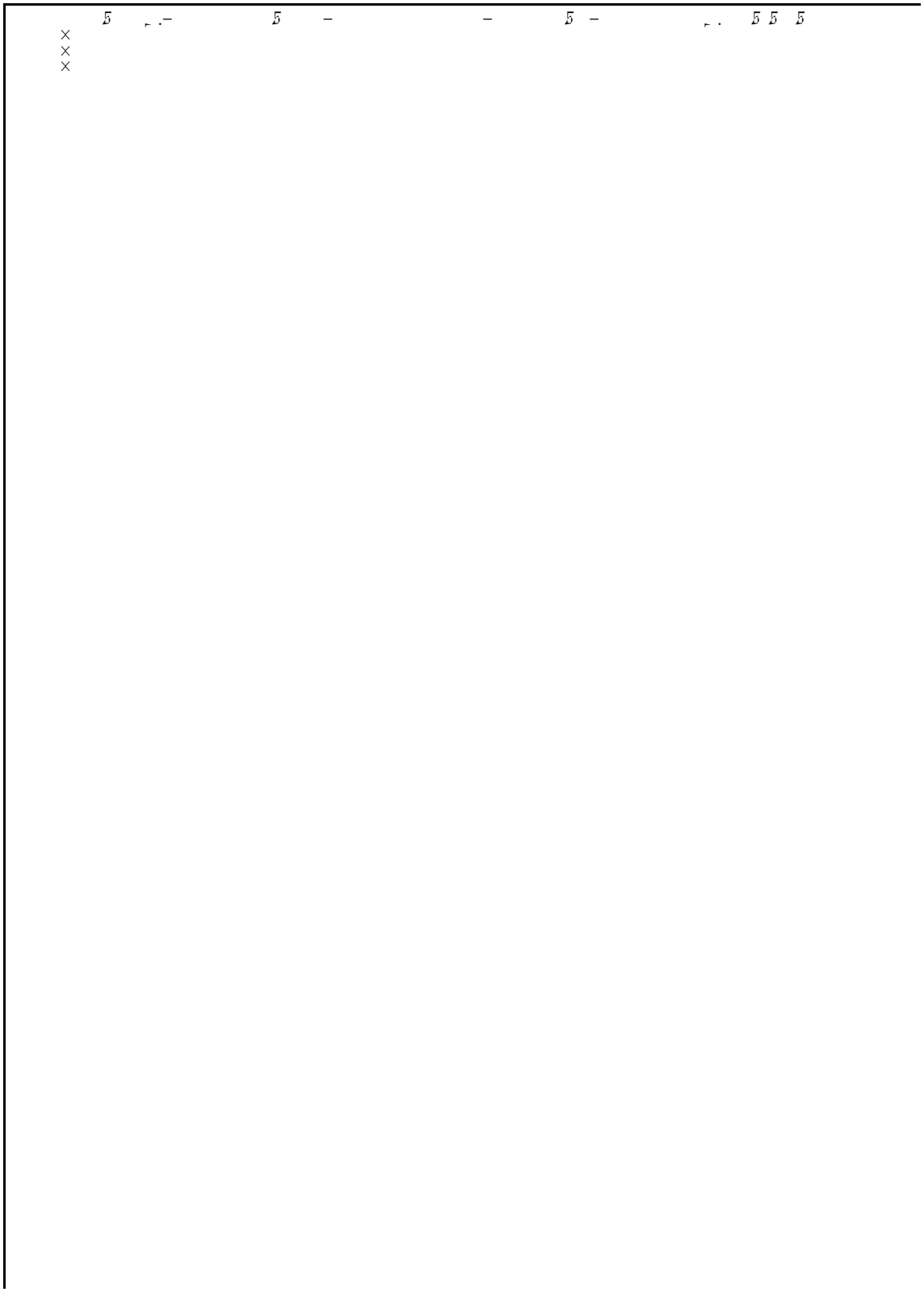
6
7
8
9

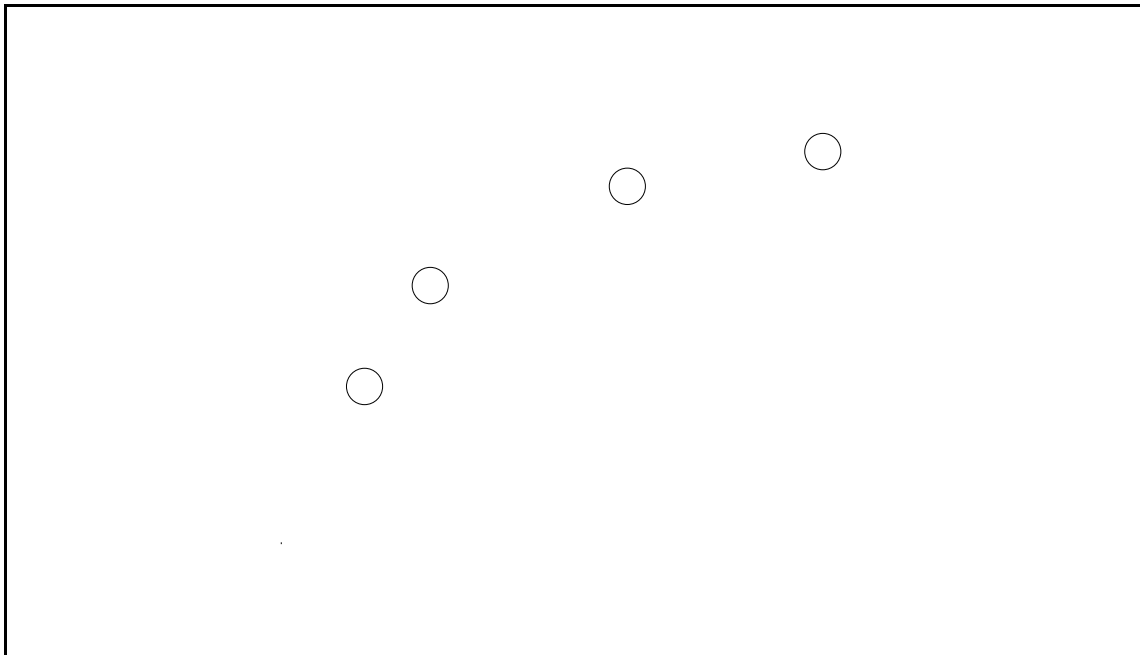


C C C C C
C C C C C

- C C C C C C C C
C C C × − C
C C C × −

- C C C C C C C C C C
C C C C C C C $a \times b$
C C C C C $a \pm 2) \times b$ C $a \times (b \pm 2)$
× C C C C

- C C C C C C 5 C
5 C CC C C C



C

Rule based processing

C C C – × C ×

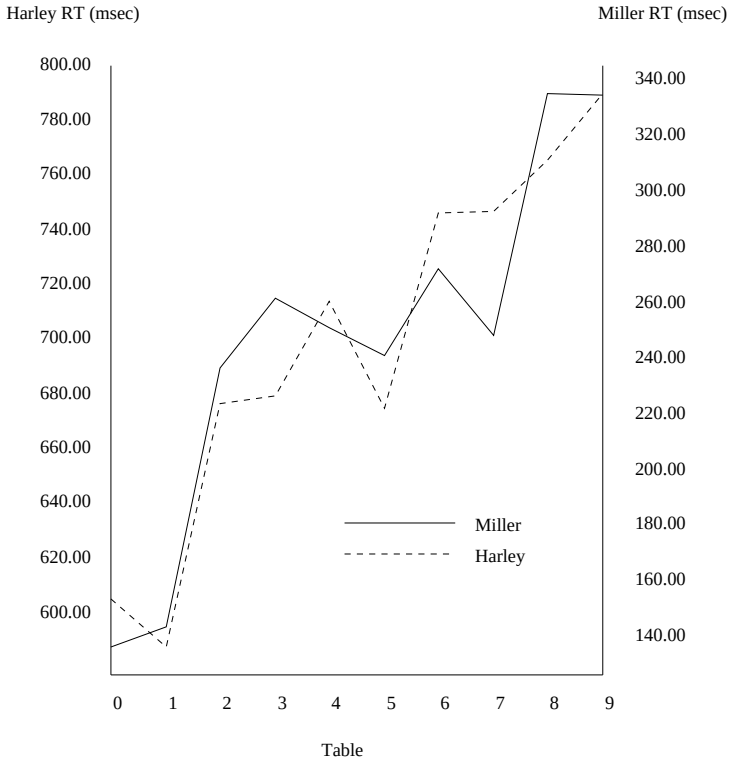
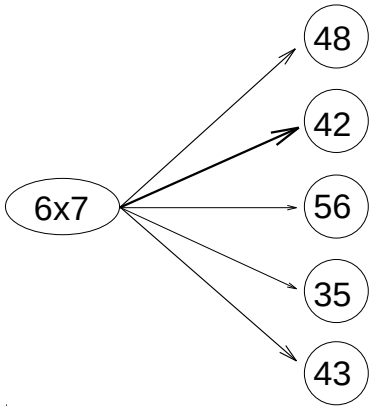


Figure C C C C C C



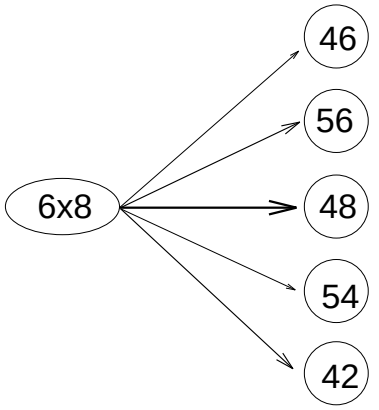
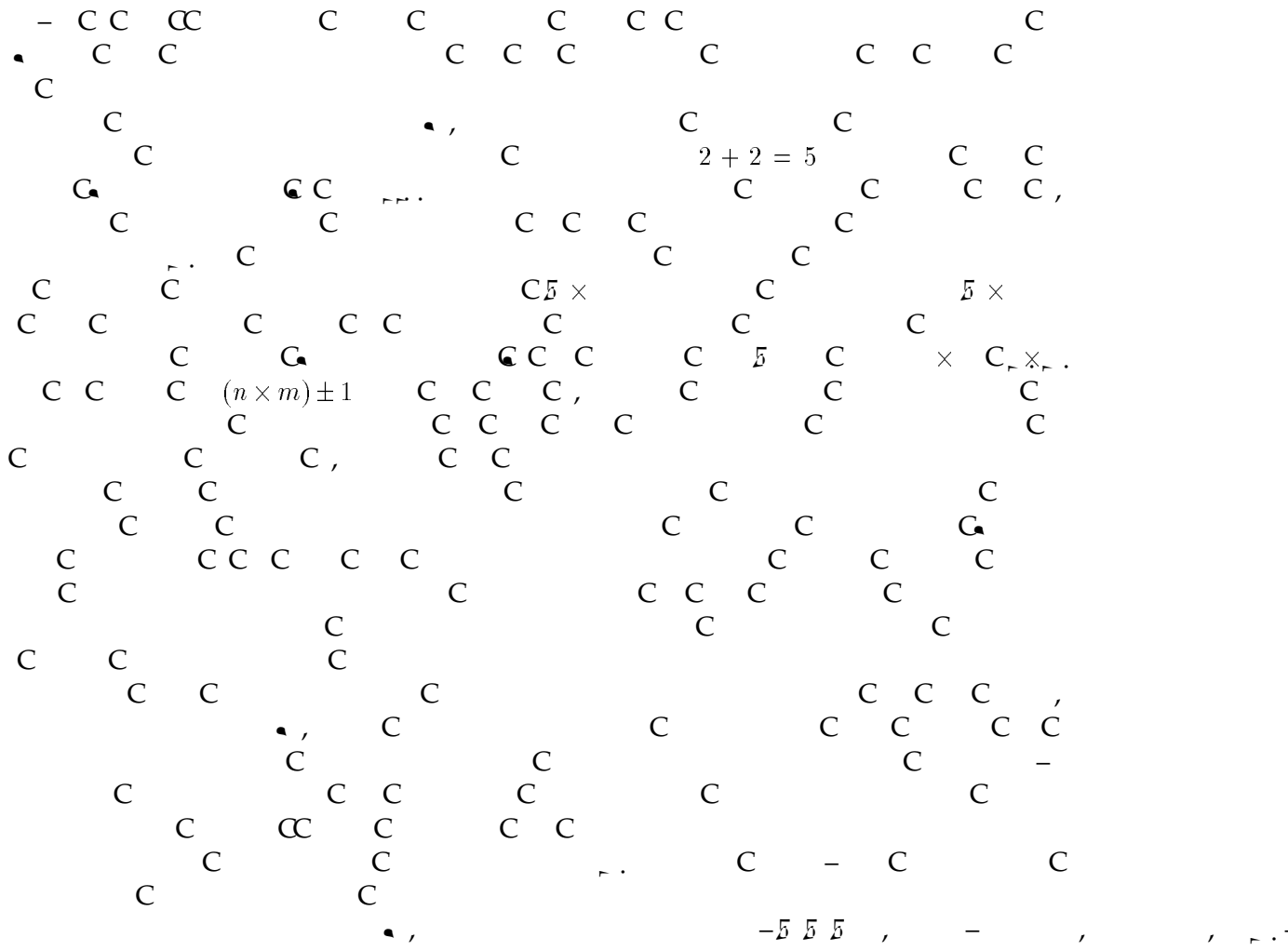


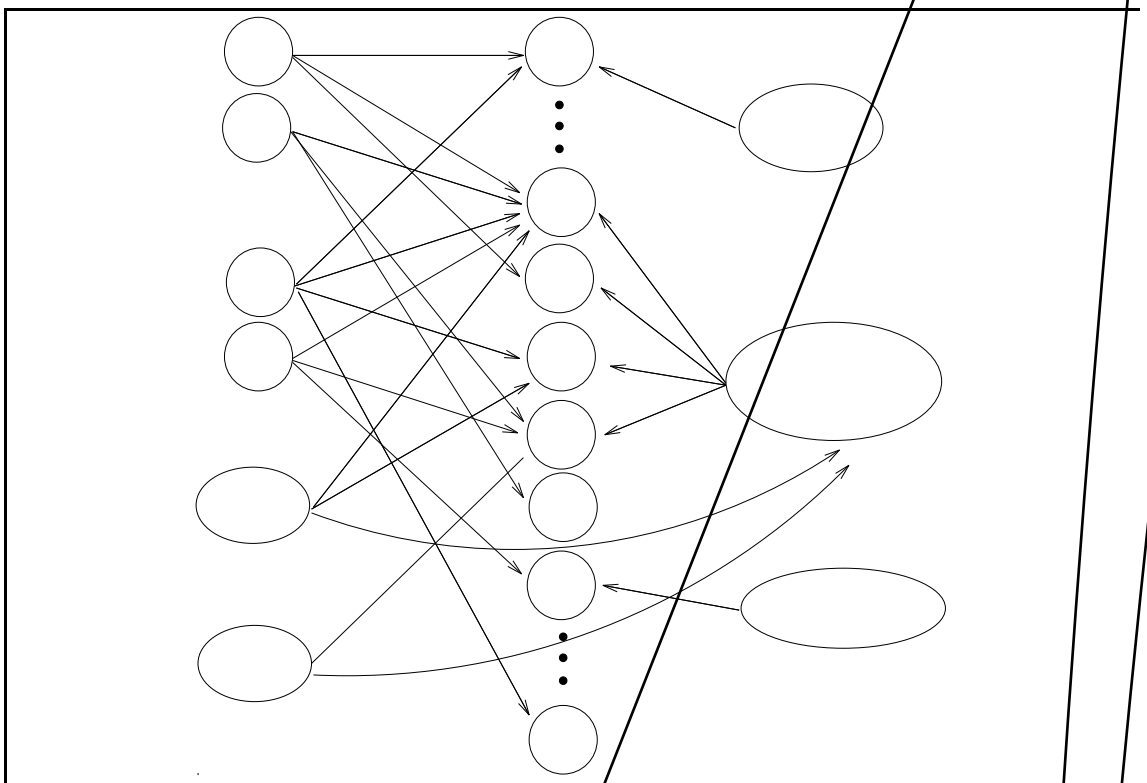
Figure C , C, C C C

C C C C C
C C C ,
C C $n \ m$
C C $n \times m$

Reaction ti e

C C C C C C
C C C C C C





Errors

Retrieva

Discussion

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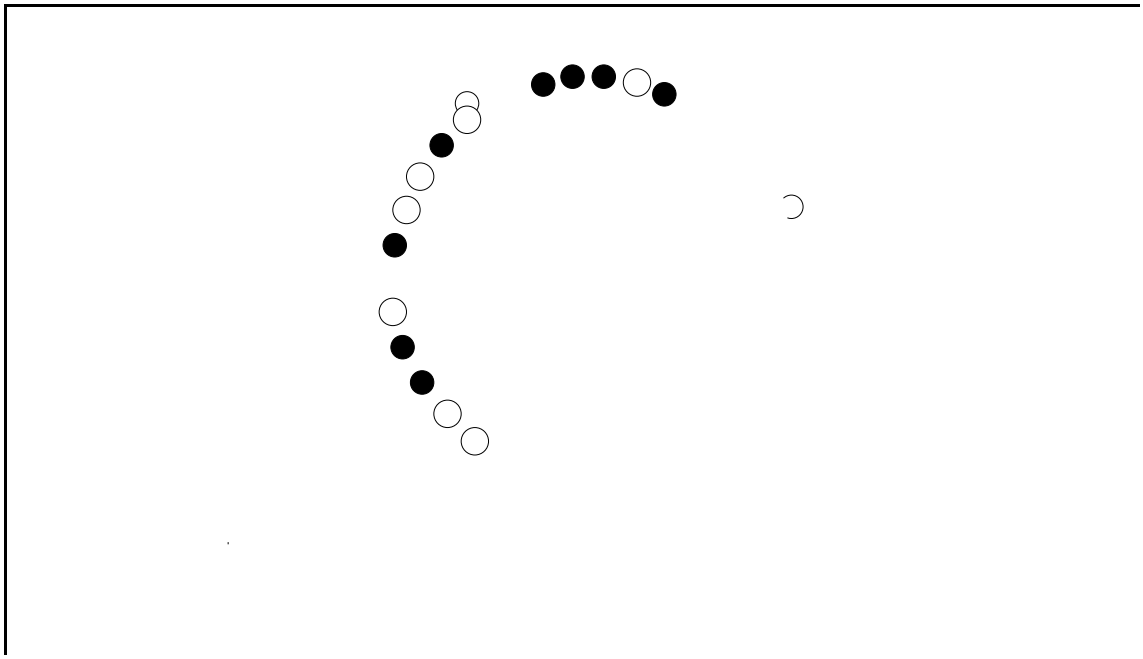


Reca

C C

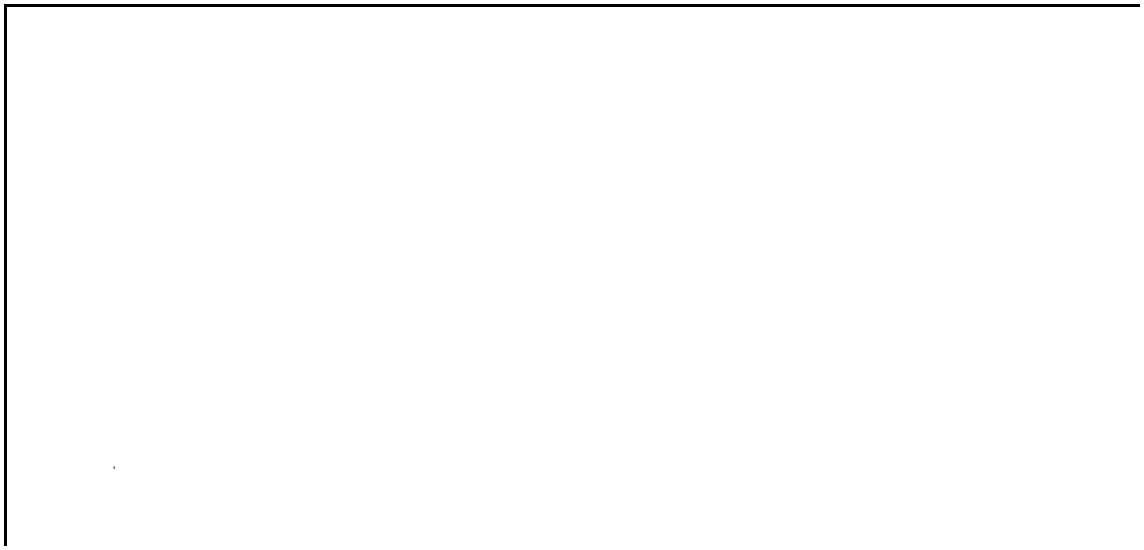
C C C

$$o_i(t+1) = \alpha \sum_j w_{ij} o_j(t) + \gamma o_i(t) + \delta f_i(0)$$

$o_i(t)$ C C i t C -1 $+1$

α C γ C $f_i(0)$ C

C δ



C C C C C– C C C C C C– C C C C C C C

Response ti es

C C , C , C, C C C C C C C C C C C C – C C C C C C C C C C C C

6 + C C 1.0 - –1.0 C

C

C C C 5

C 5

C C – –

C , C 5

Tab,e C C C C

C C C , C

C C C C C
C C C C C C C ,
C C C C ,
C C C
C C , C C C C

Da age

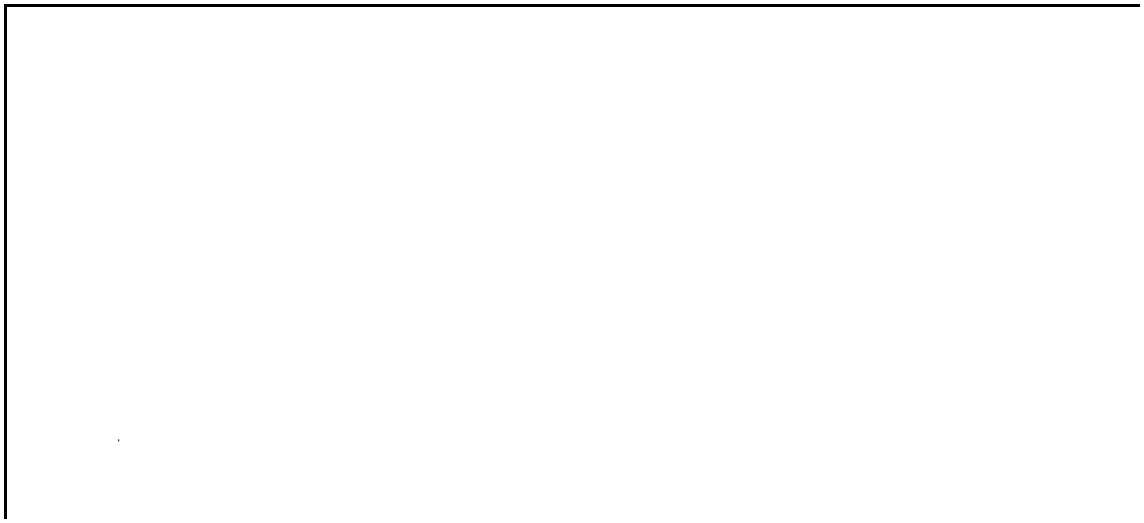
C , C
C C C C
C C C C
C C C C C
C C – C
C C – 5
C C C C . .
C , C C C

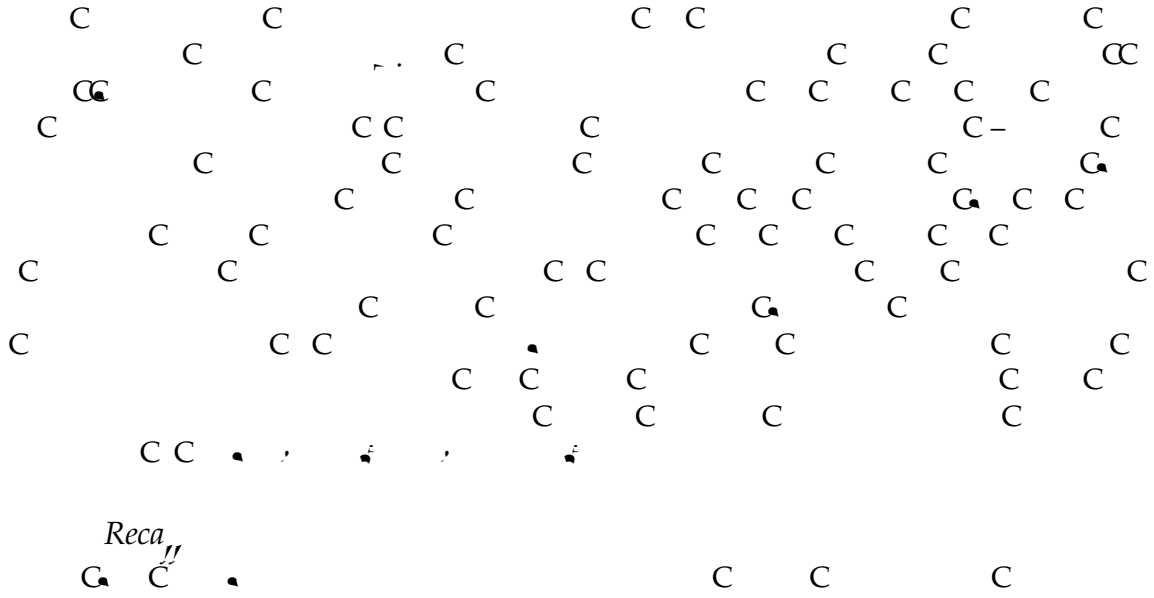
C C C C CC Ga
C Ga C C C
C C C C C C C

- C , C C C C Ga
C C C C C
C C C C C C
C C

- C C
C C C C
C C C CC

- C C
C C C C C
C C C
, ,6 C ,6 ,6 Ga C Ga





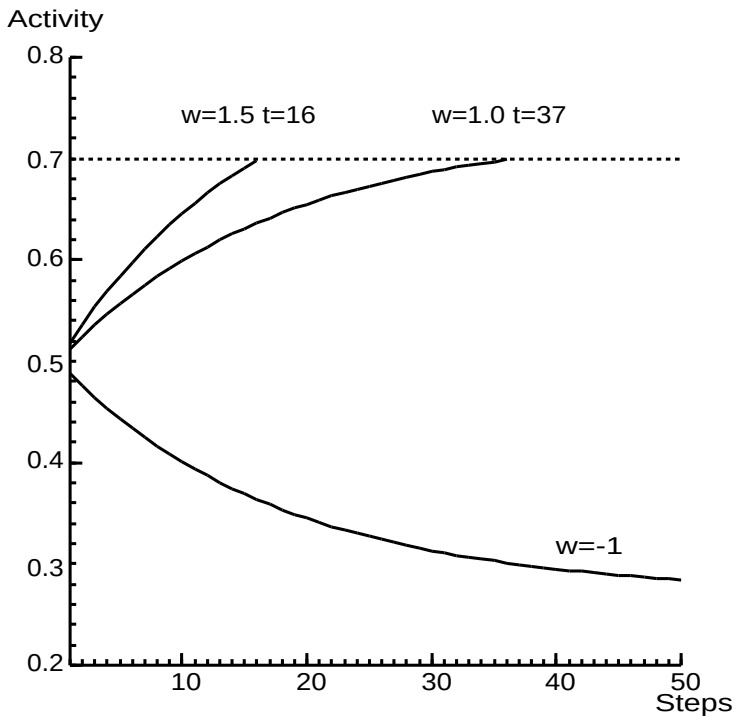
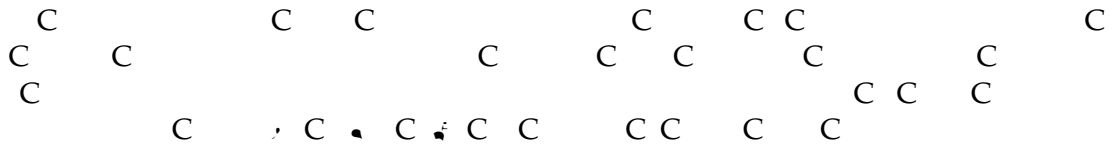


Figure C C C C

C C C C C C

$$o_i = \frac{a_i}{\sum_j a_j}$$

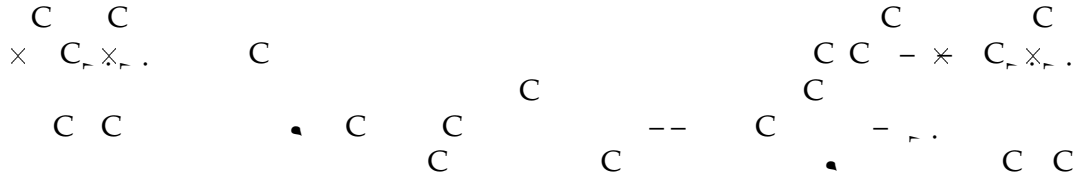
i j C C C C C C
C C C C C C C
C C C C C C
C C C C C C C
C C C C C C C C
C C C C C C C
C C C C C C C



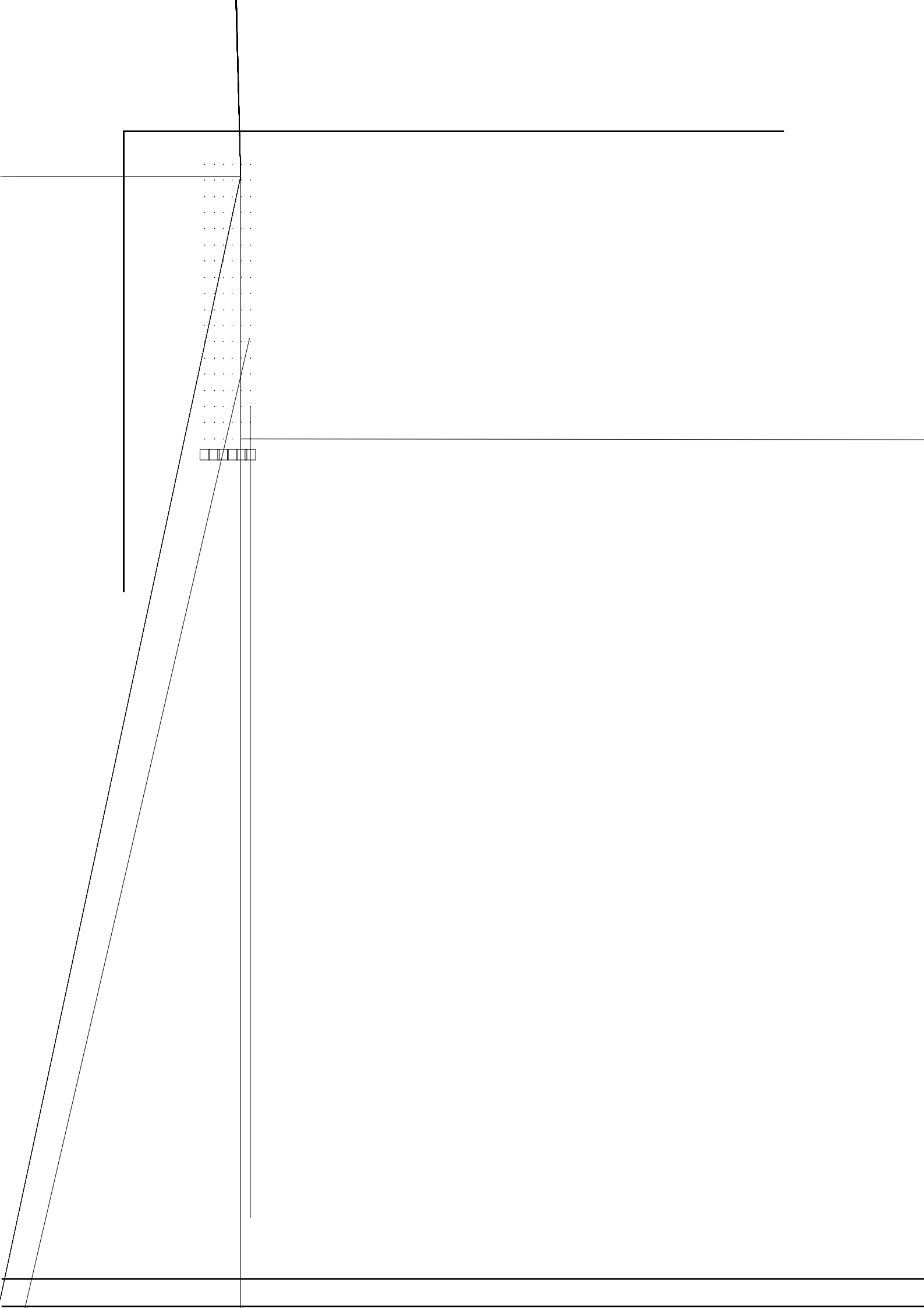
C C C C C C C
C C C C C C C
C C C C
C , C . C : C C C C C C C

Training

C C C C
× C ×. C C C – × C ×.
C C
C C . C C – – C – .
C C . C C



C C C
C C ,
C C C C
C C C C C
C C C C C



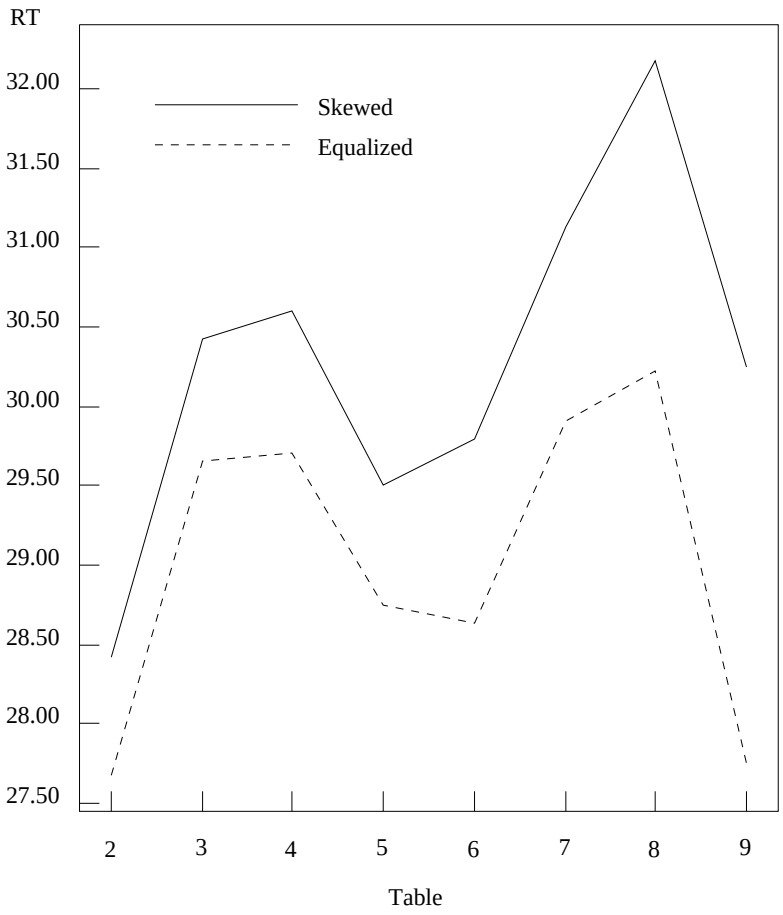
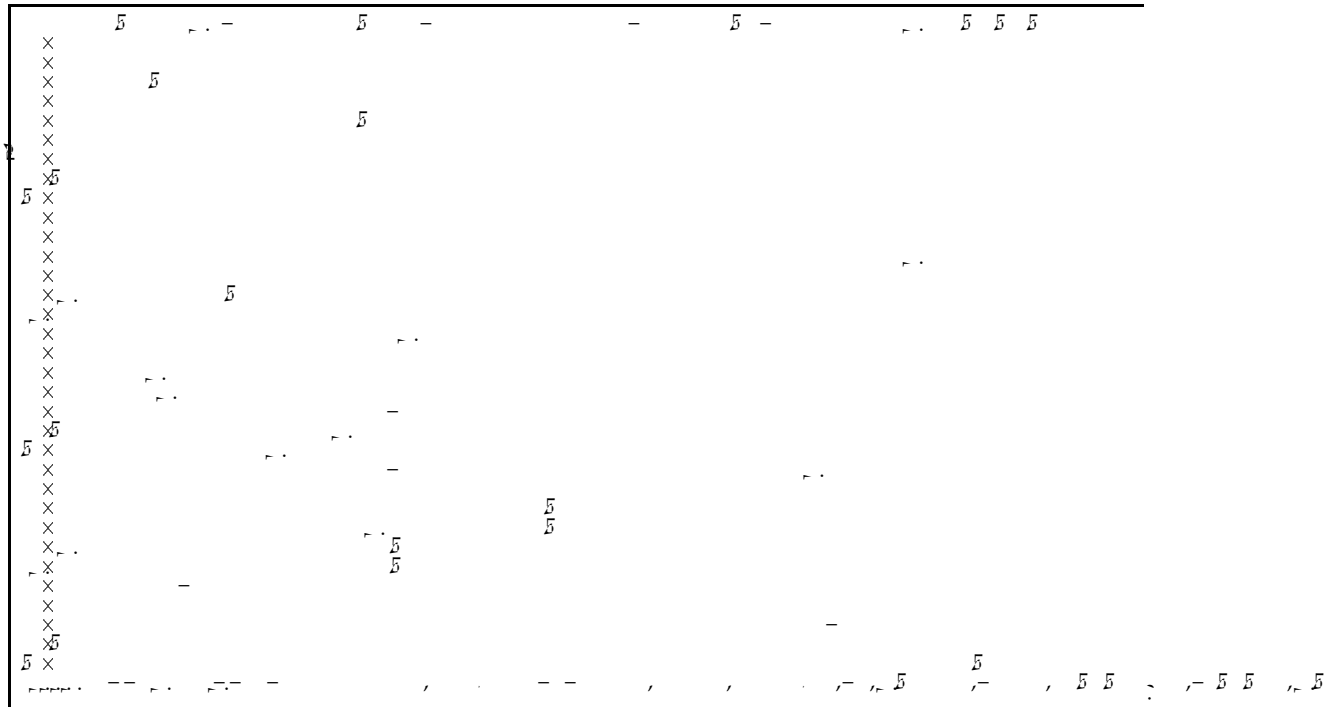
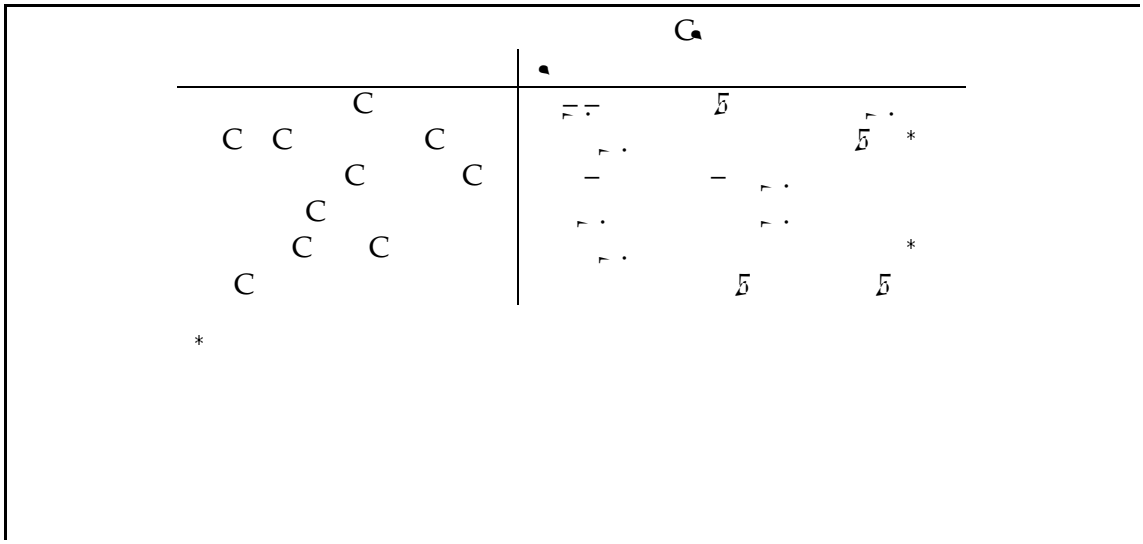


Figure C C C C C
C C C – G –

C C C C
C C G C5 C C C C
C C C G 5 ×5
C C
C C C C – G C
C C C C





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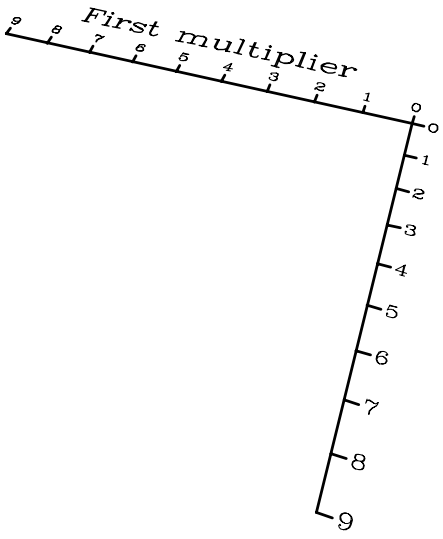
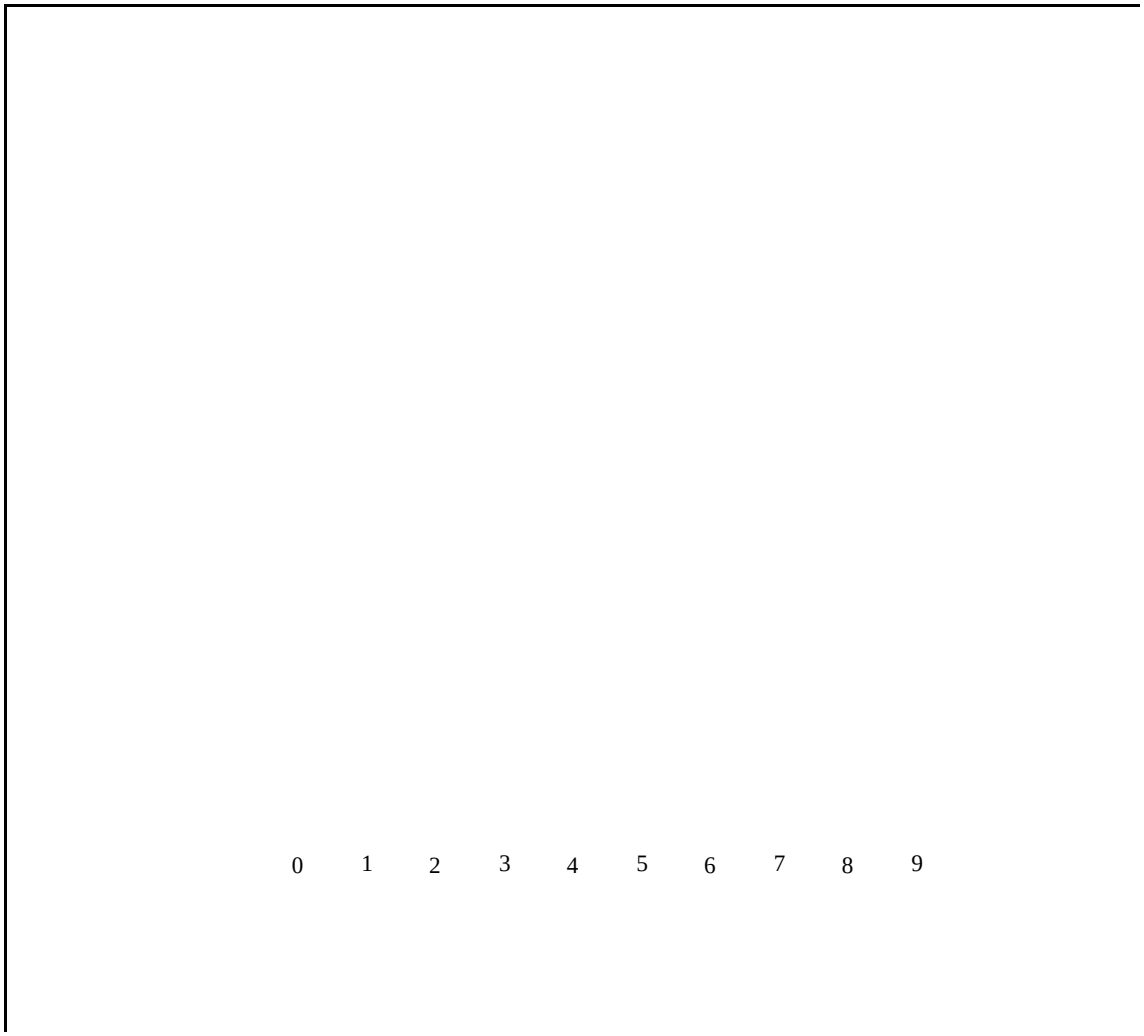
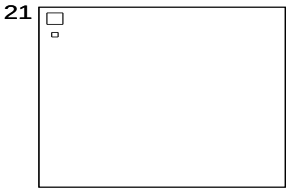

First multiplier
9 8 7 6 5 4 3 2 1 0
0 1 2 3 4 5 6 7 8 9


0 1 2 3 4 5 6 7 8 9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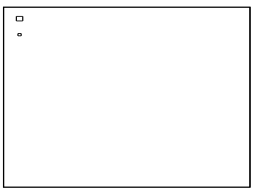
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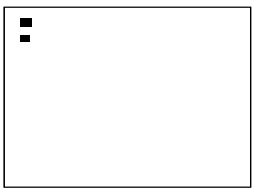
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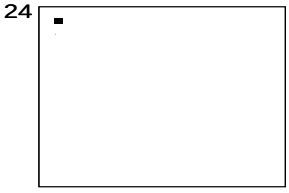
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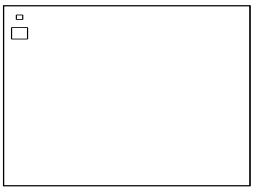
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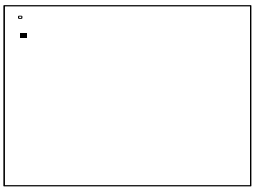
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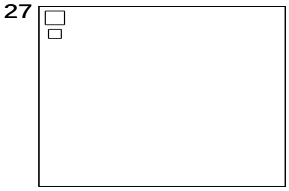
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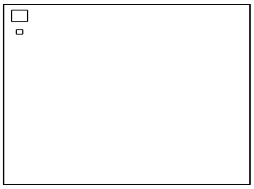
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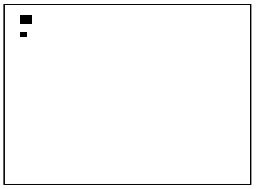
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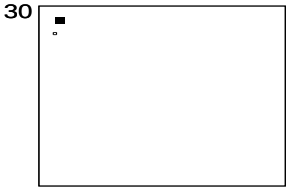
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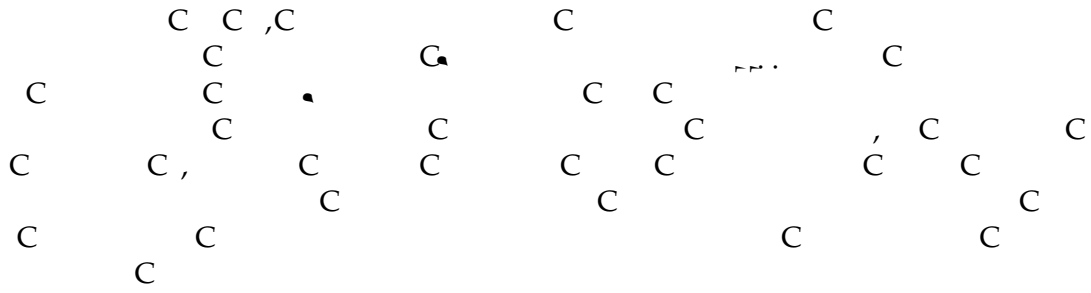

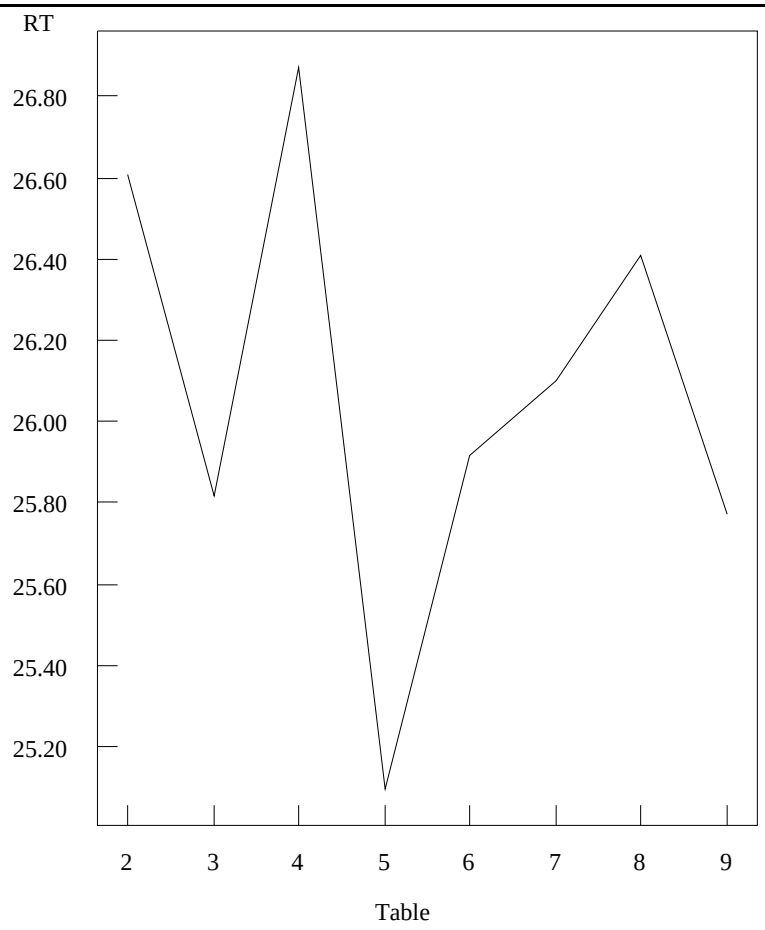

Figure C C C – G C

C , C • C C – C

C C C

• C C C C

• C C – × C × . • C –

C C– C – × – C C– C

× C C

$$\text{frequency} = \frac{180 - \text{product}}{360}$$

• C • C

C C C C C – . , .

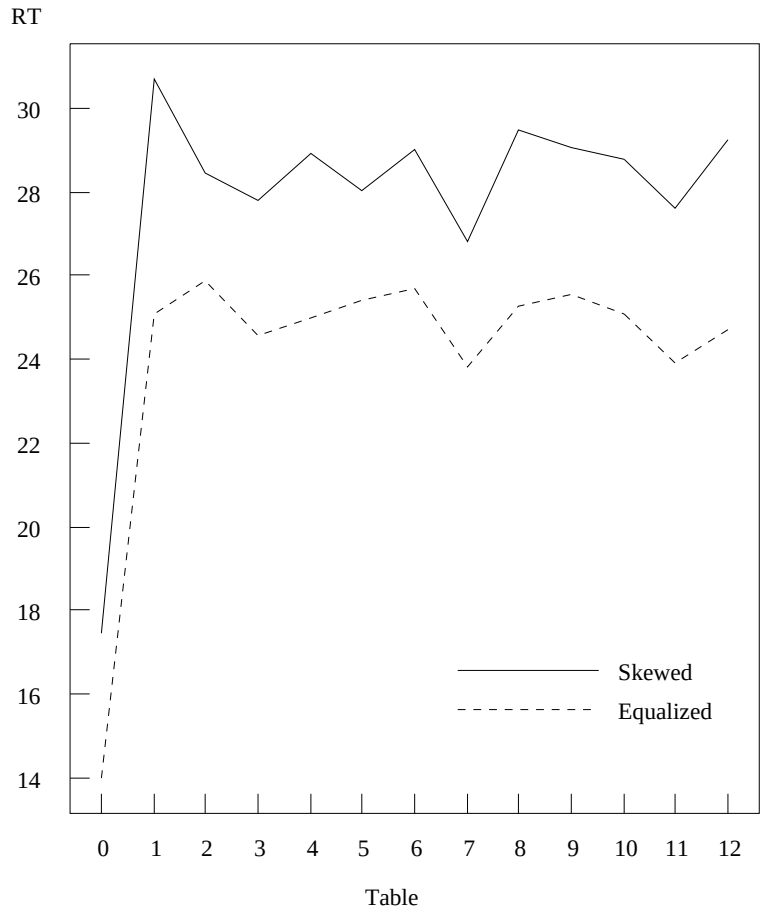

Figure C C –

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5	–	–	–	–		—	—	–	–	–
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	–	–	–	—	–	5.	–	—	—	
	–	—	—	–	–	—	–	–	–	–

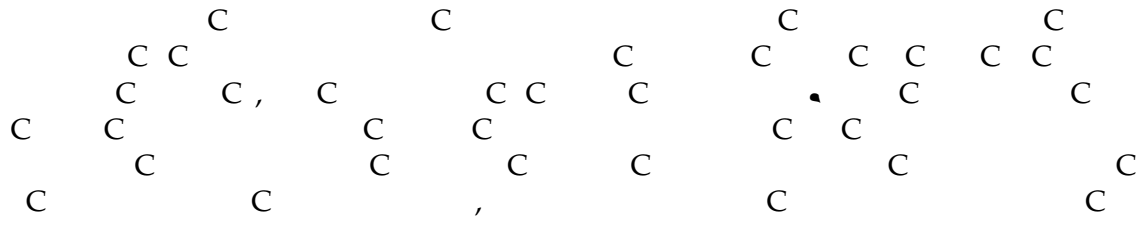
Tab e C C C C – × C × . C C

C C , .

C C , C .

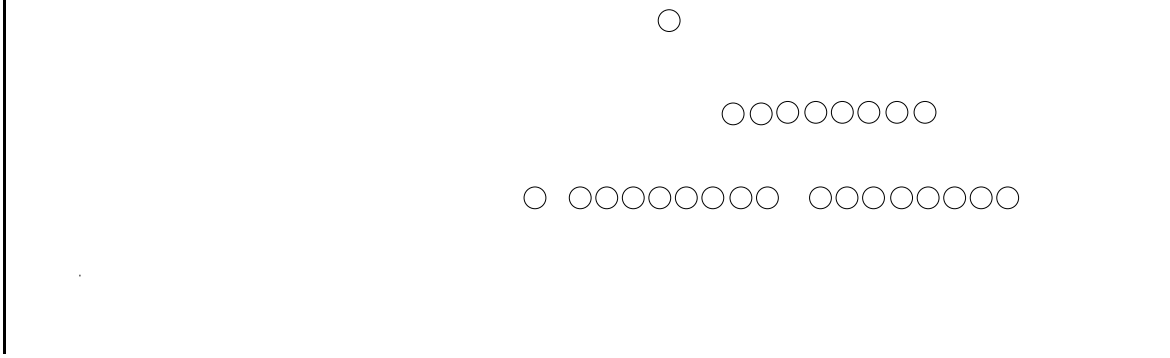
C , .

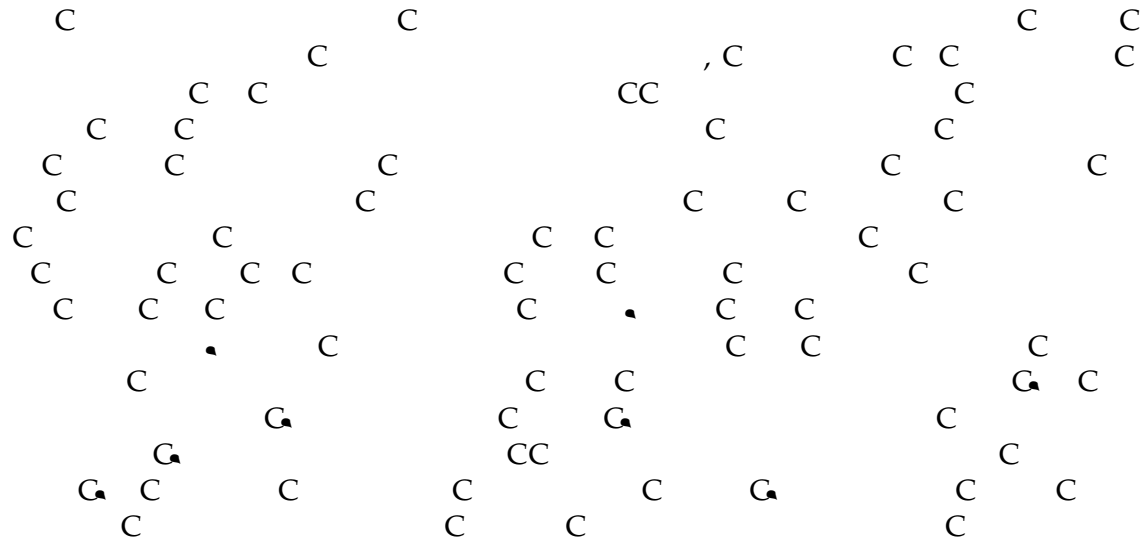
5



Choice of output representation

True False





81
72
64
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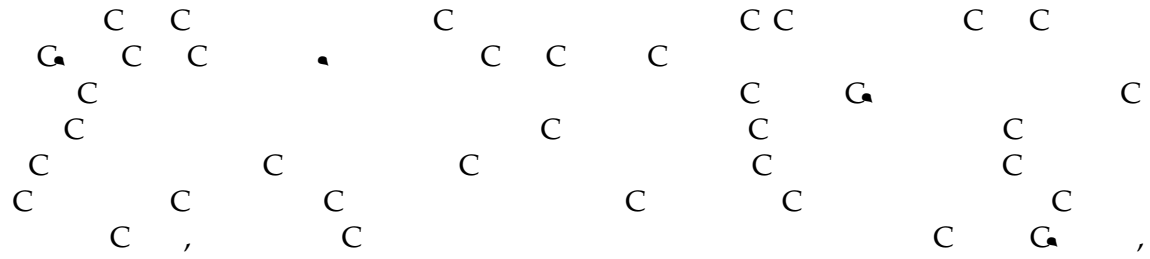
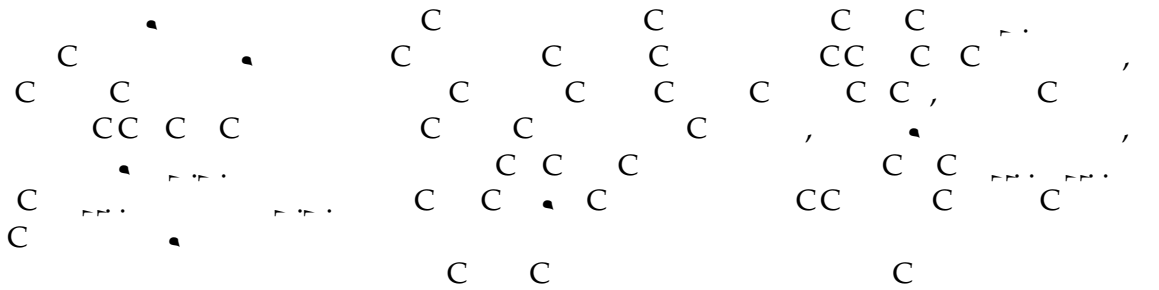
?
3x3



Mu tico u n Arith etic

Sy bo ic Accounts of Arith etic

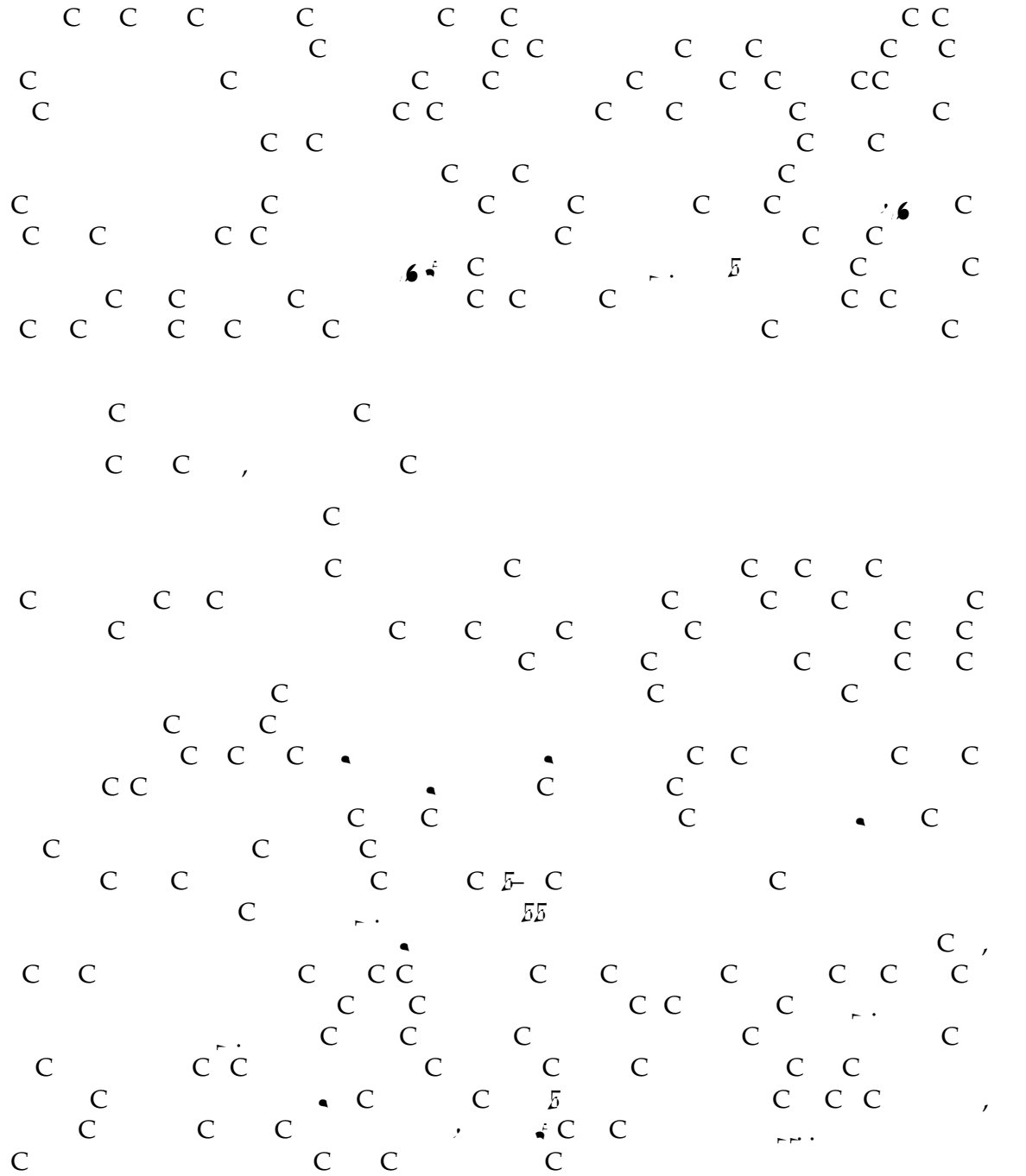


	Conditions		*Actions*
INTO:	[processmult]	⇒	readintandb();
SM:	[t ?t] [b ?b] [c ?c]	⇒	do_calc();
NX:	[next_top]	⇒	[processmult] shift_top_left();
WM:	[result ?u] [carry ?c]	⇒	writedown(); [next_top]
CC:	[no_more_top]	⇒	checkcarry(); [checkbottom] [addzero]
CB:	[checkbottom]	⇒	check_bottom();
FI:	[none_left]	⇒	[stop]
NB:	[no_more]	⇒	endmult(); [startadd]
CO:	[startadd]	⇒	readincolumn();
DA:	[column ?len ?dig]	⇒	do_add();
ML:	[next_left]	⇒	[startadd] moveleft();
WA:	[u ?u] [c ?c]	⇒	writeadd(); [next_left]
CA:	[no_more_digits]	⇒	checkadd();
AZ:	[addzero]	⇒	add_zero();

Tab e

4.2 Models

[processmult] C C C C C
C C C
C C C C



Learning by induction

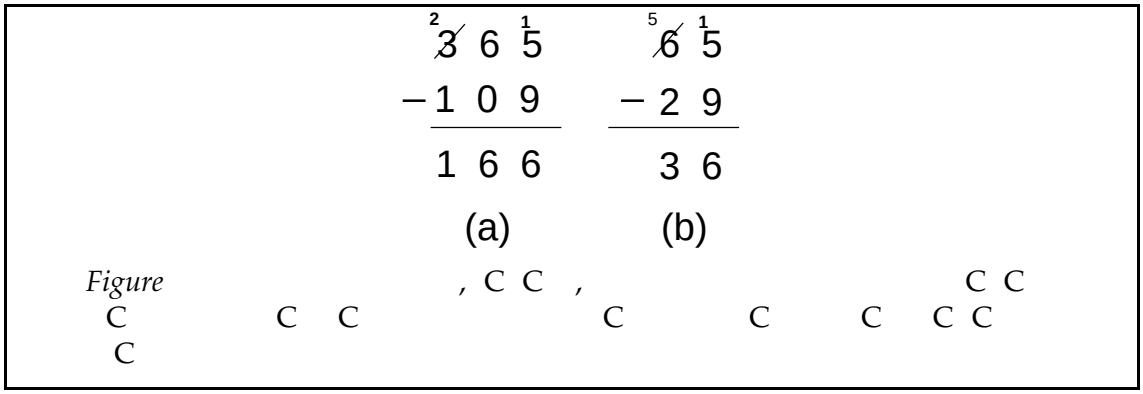
```
(Problem 1)                  Object   is a subtraction prob e
(Column 2) (Column 3)        Objects        are co u   ns
(Part 1 2) (Part 1 3)        The co u   ns are part of the prob e
(First 1 3)                  Object   is the eft   ost object
(Adjacent 1 2 3)             The co u   ns are adjacent
(Cell 4) (Cell 5) (Cell 6)   Objects   _  are ce s
(Digit 4) (Digit 5)          Objects   and   are digits
(Blank 6)                    Object   is a b an   ce
```

```
Sub1Col(C) OR
1. [And (Digit T) (Part-of T C) (First T C)
        (Digit B) (Part-of B C) (Middle B C)
        (Ordered C T B) (Adjacent C T B)
        (Value-of TV T) (Value-of BV B) (LessThan TV BV)
   ->   (Borrow C)

2. [And (Digit T) (Part-of T C) (First T C)
        (Digit B) (Part-of B C) (Middle B C)
        (Ordered C T B) (Adjacent C TB)
        (Value-of TV T) (Value-of BV B)
        (Less-Than-or-Equal BV TV)
   ->   (Diff C)

Diff(c) AND
1. [And (Digit T) (Part-of T C) (First T C)
        (Digit B) (Part-of B C) (Middle B C)
        (Cell A) (Part-of A C) (Last A C)
        (Ordered C T B) (Adjacent C T B) (Ordered C BA)
        (Value-of TV T) (Value-of BV B)
        (AbsoluteDifference TV BV AV)
   ->   (Write AV A)
```

Tab e C C C C – . C ,
C C C C C .
–

The i passe repair process

C C C C C C C C C
C C C C
C C C C C C
C C C C C C C C C



Figure

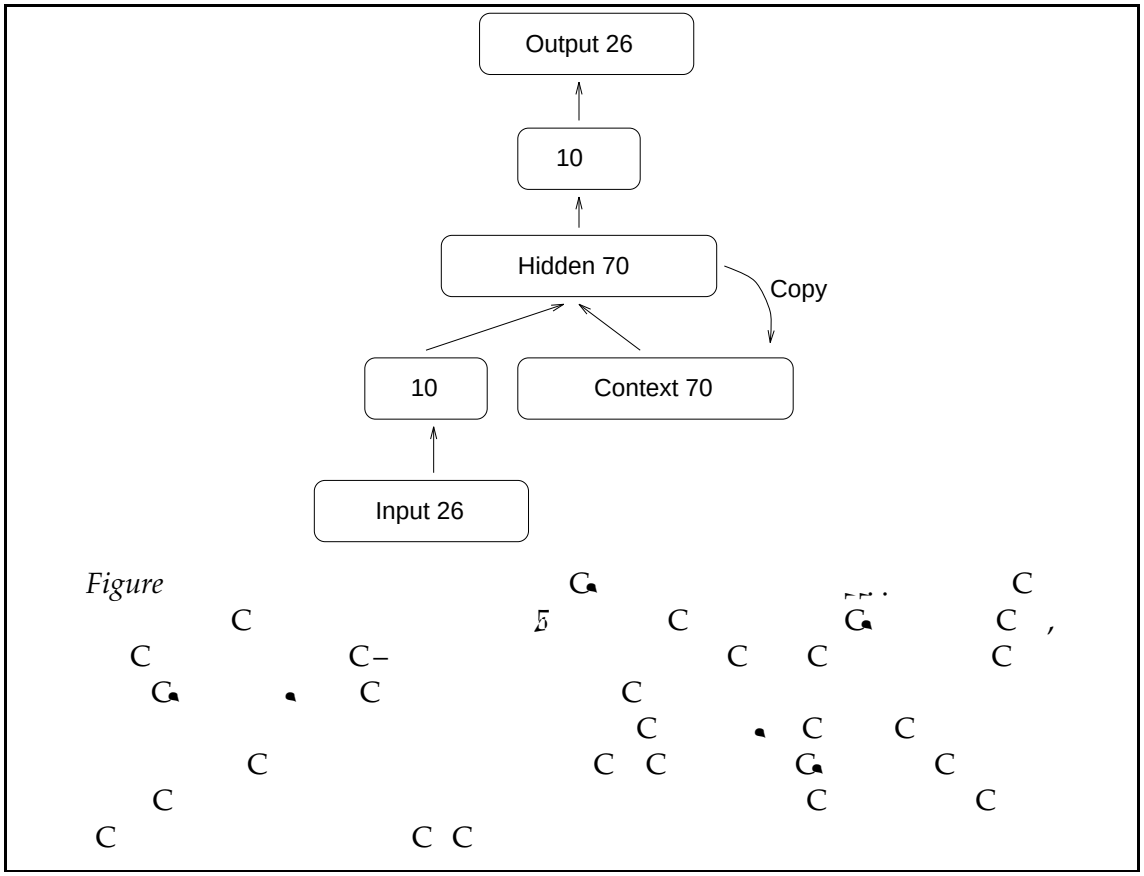
E pirica, adequacy

C C C C
C C C C C *patch* C
C C C
C C C
C C
C C C C C
C C C C
C C

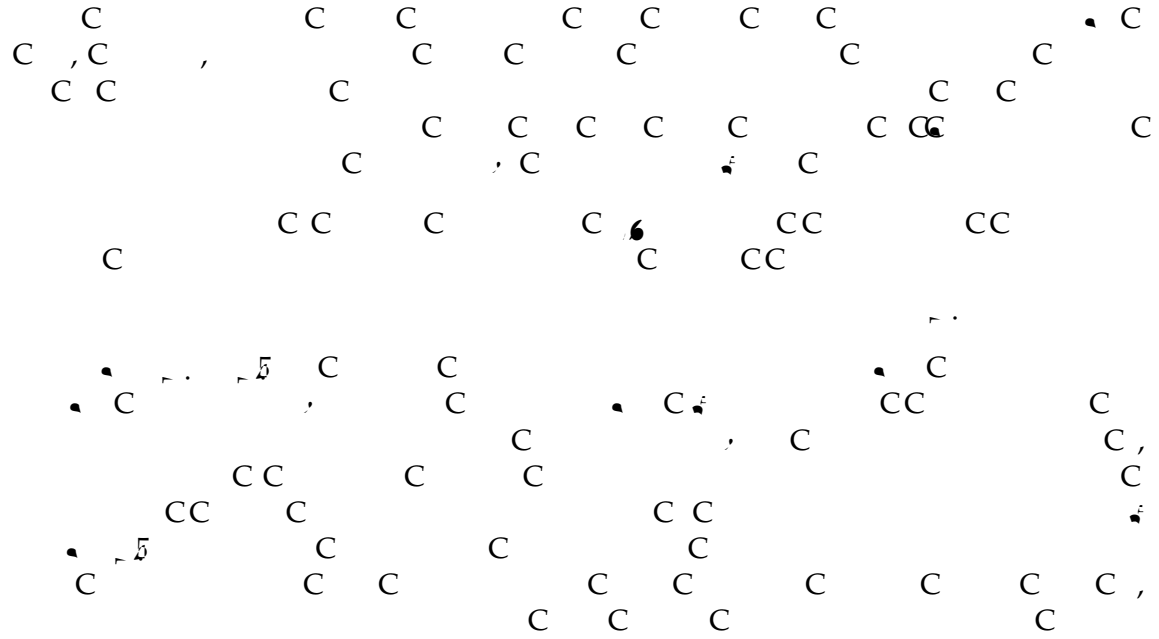
C C C C
C C C C
C C ,
C C , C C C C
C C C C C ,
C C C C C
C C C C
C C C ,
C C C C C C
C C C
C C ,
C C C C C C

C C C

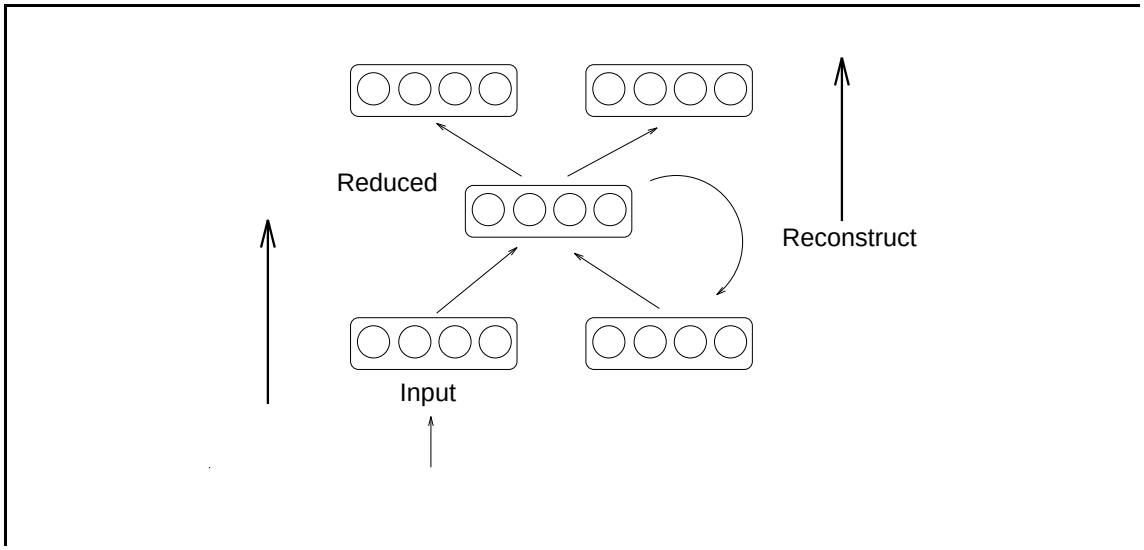
Hidden unit recruit ent



Figure

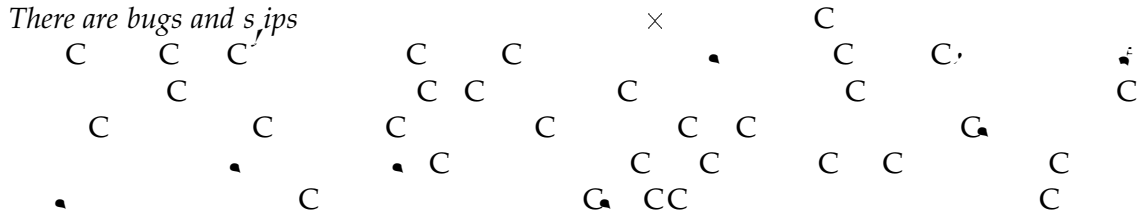


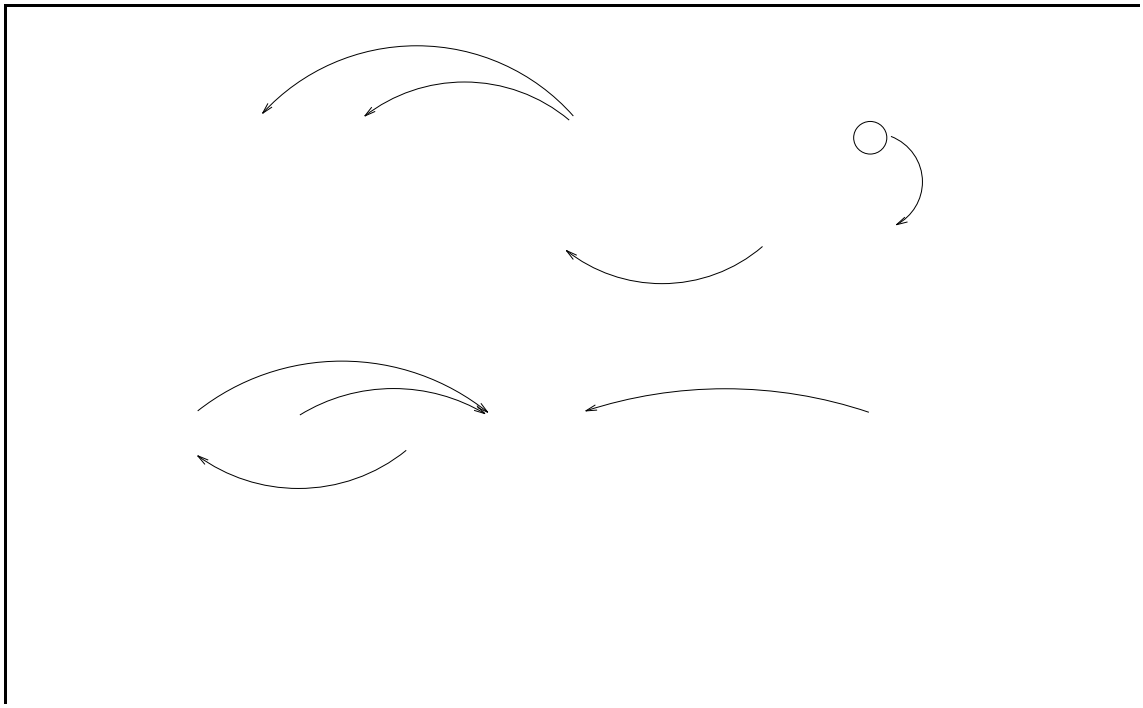
C C , C C
C ,
C C C C

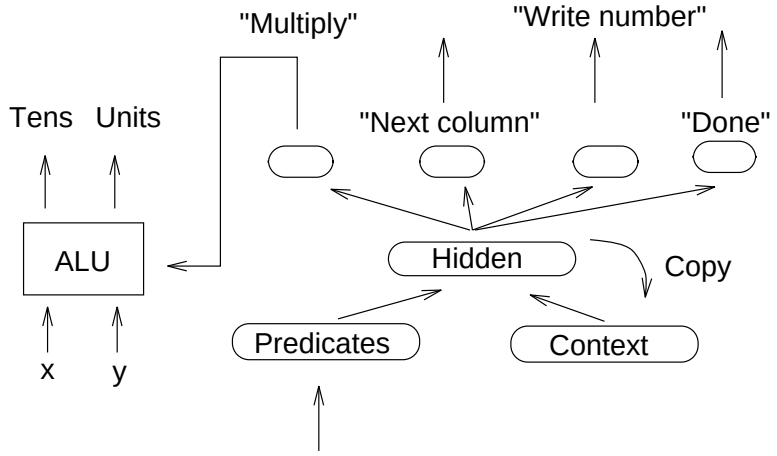

Reduced
Reconstruct
Input

C C C C
C C C C
C
, C C C C C
C C C C C

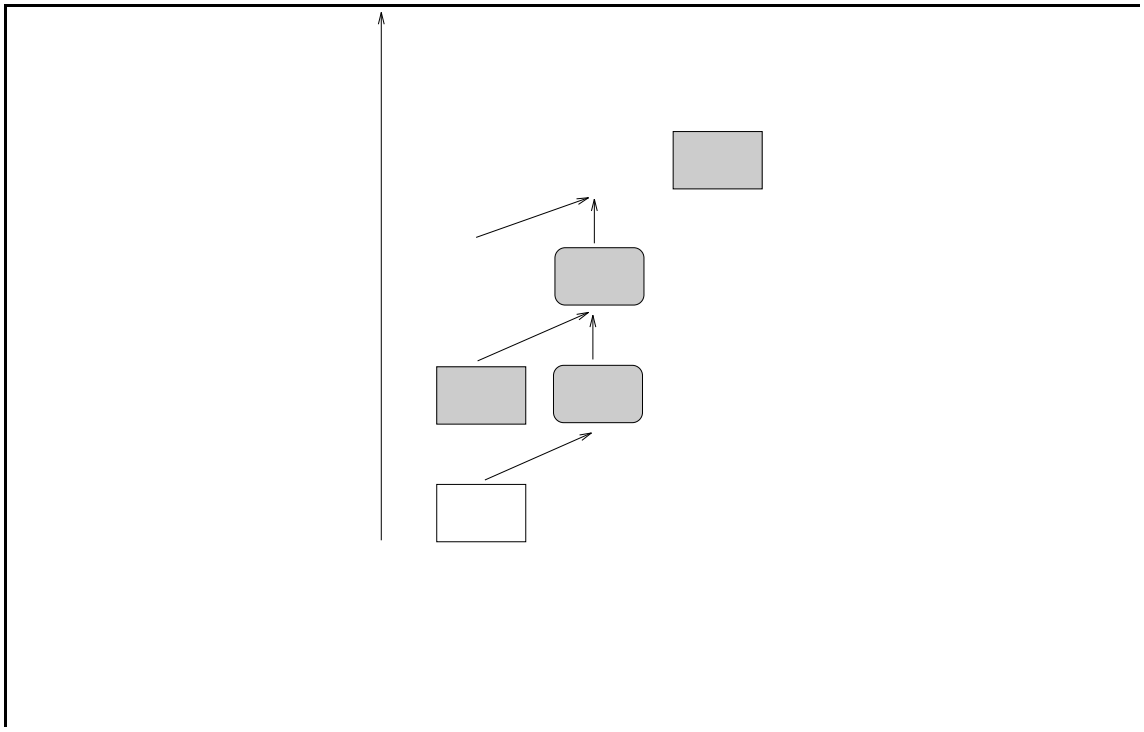
C C C C C CC
C C C C C
C C C C
C C C C C ,
C C C C C
C C C
C C C C C C
C C C C C C


There are bugs and s ips




"Multiply"
"Write number"
Tens
Units
"Next column"
"Done"
ALU
Hidden
Copy
Predicates
Context
x
y

Figure



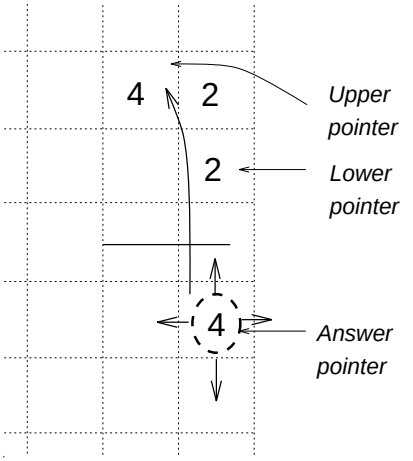


Figure C C

top_next_column (TNC)	store_mark (STR)
ump_answer_space (JAS)	zero_accumulator (ZAC)
ump_top_row (JTR)	next_answer_row (NAR)
left (LFT)	next_bottom_column (NBC)
right (RHT)	inc_answer_column (IAC)
up (UP_)	inc_top_column (ITC)
down (DWN)	add_start_position (SAD)
read_carry (RDC)	start_multiplication (SMU)

write_units (UNI)	add_mark_to_accumulator (ADD)
write_tens (TEN)	compute_product (MUL)
mark_zero (MKZ)	draw_rule (RUL)
mark_carry (MKC)	done (DON)

Tab e

add_start_position

start_multiplication

mark_carry

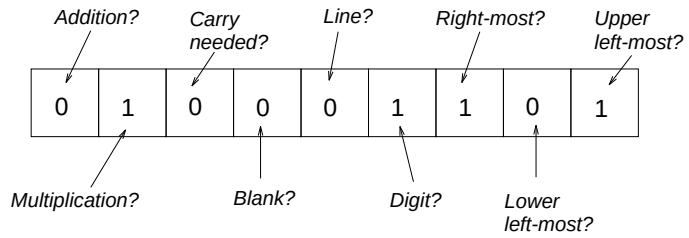
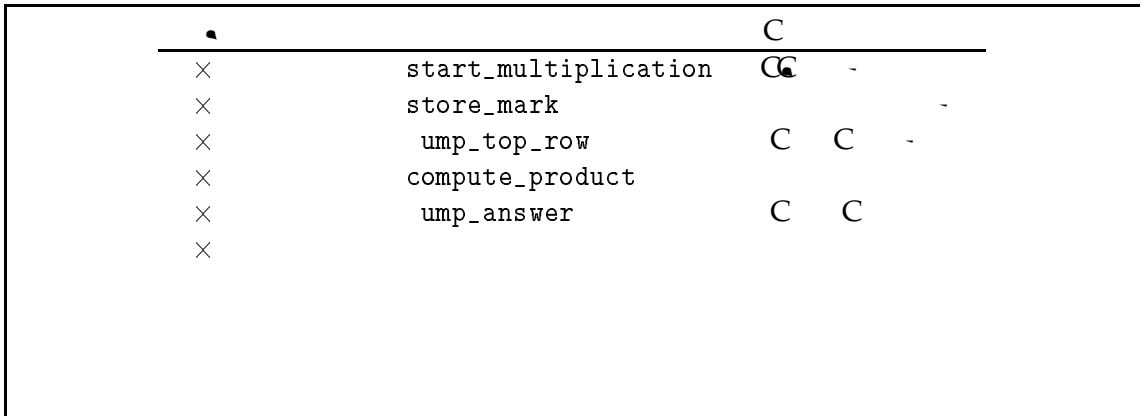
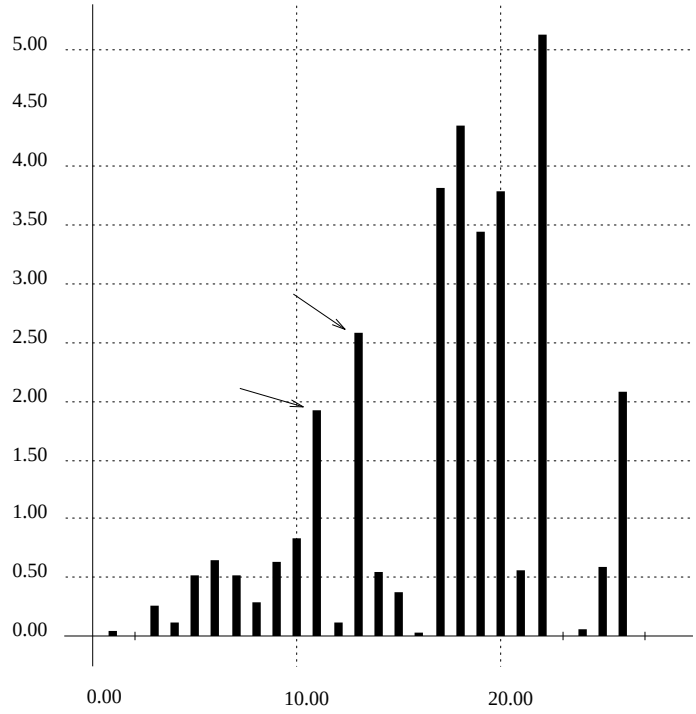

Addition?
Carry needed?
Line?
Right-most?
Upper left-most?
0
1
0
0
0
1
1
0
1
Multiplication?
Blank?
Digit?
Lower left-most?

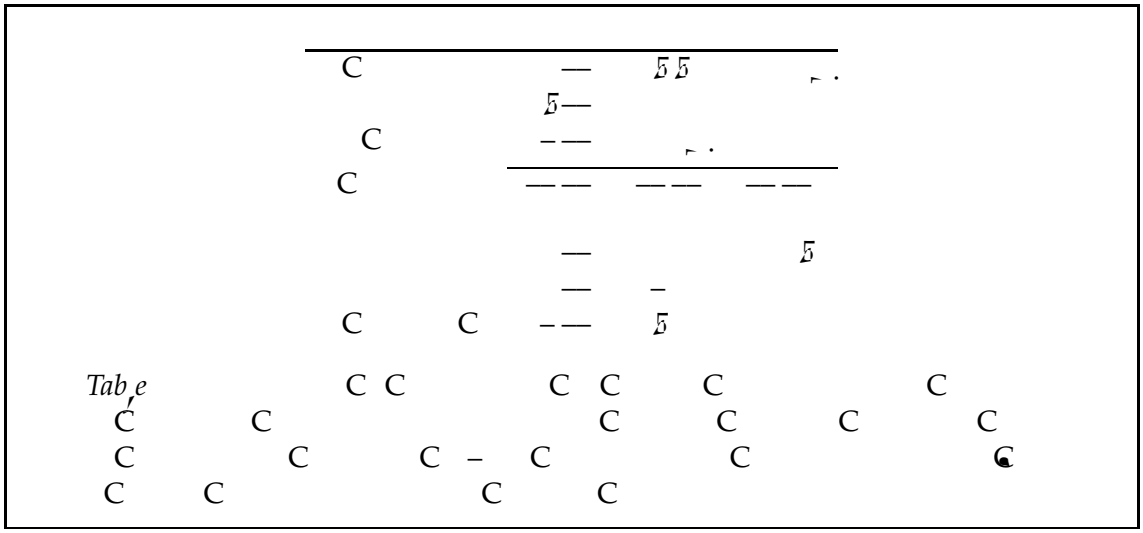
Figure C

C C C


start_multiplication
store_mark
ump_top_row
compute_product
ump_answer


5.00
4.50
4.00
3.50
3.00
2.50
2.00
1.50
1.00
0.50
0.00
0.00
10.00
20.00

C C C
C C C C
C C C
1 1 1
× 1 1



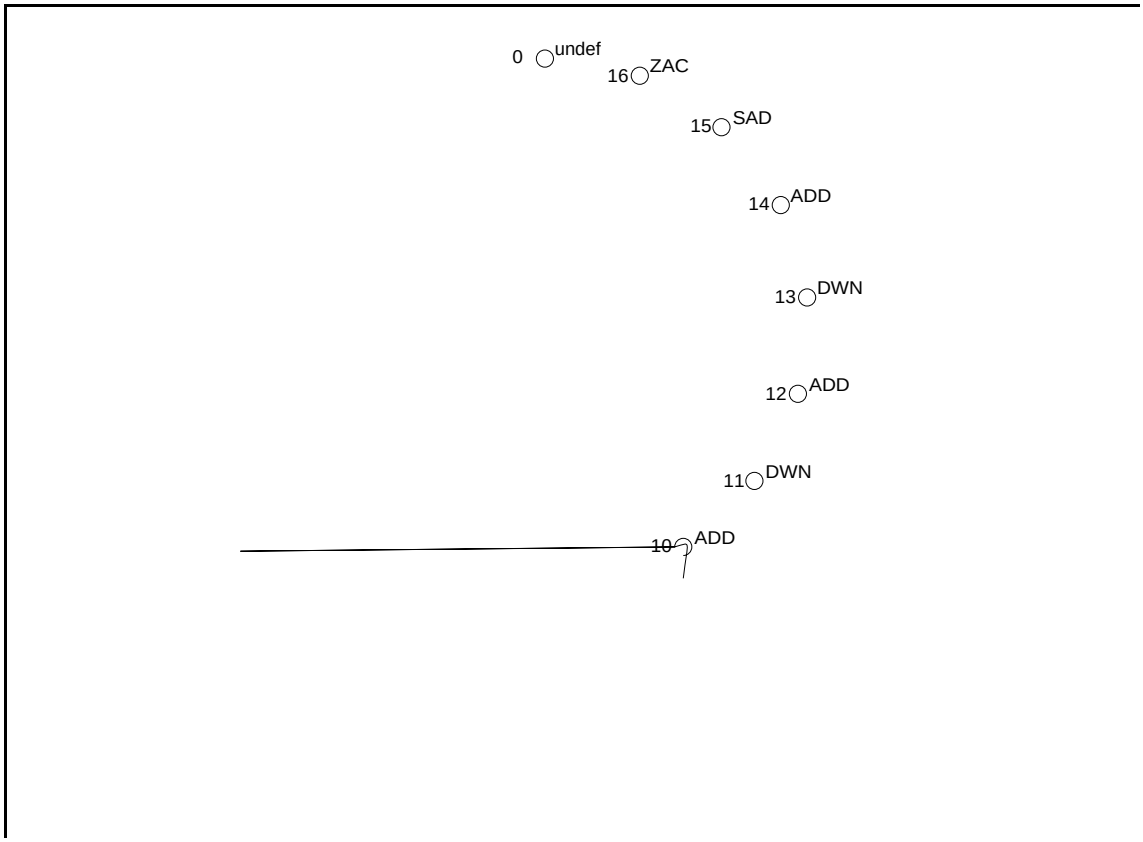
Correct behaviour

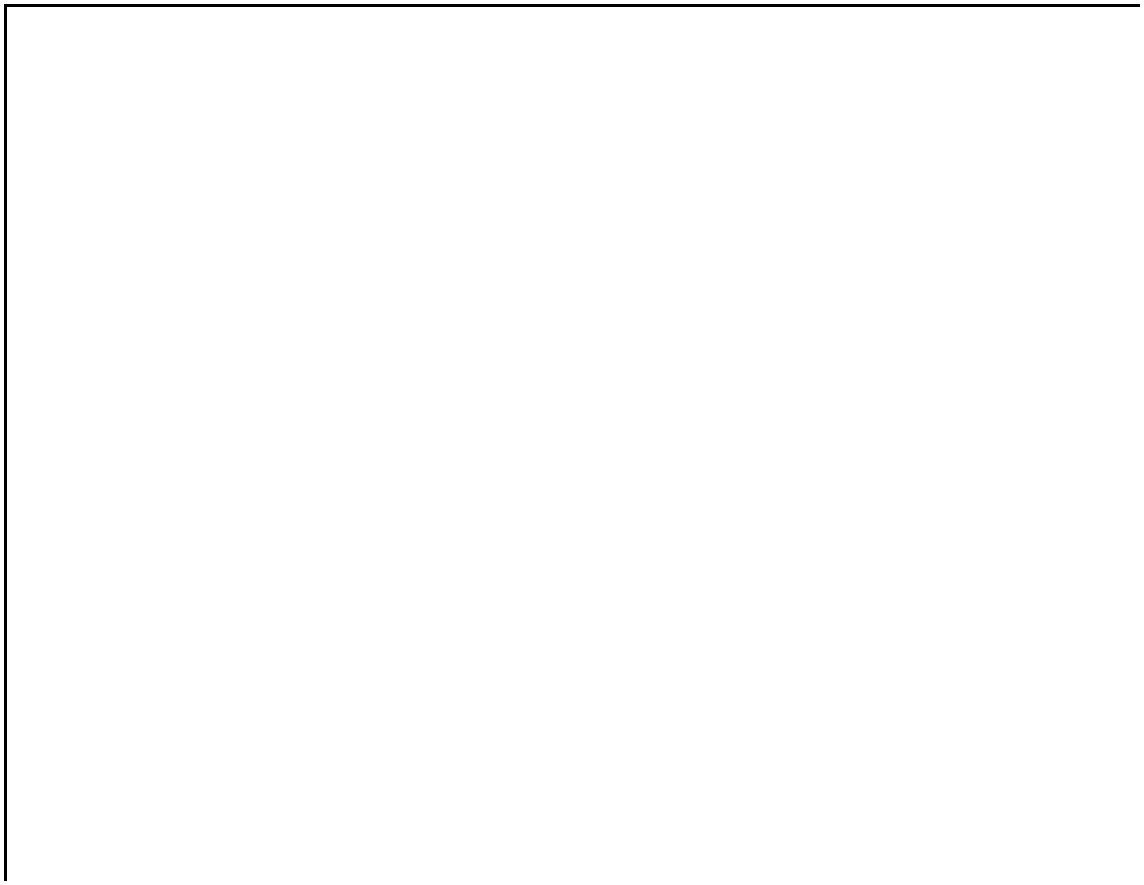
Star bugs

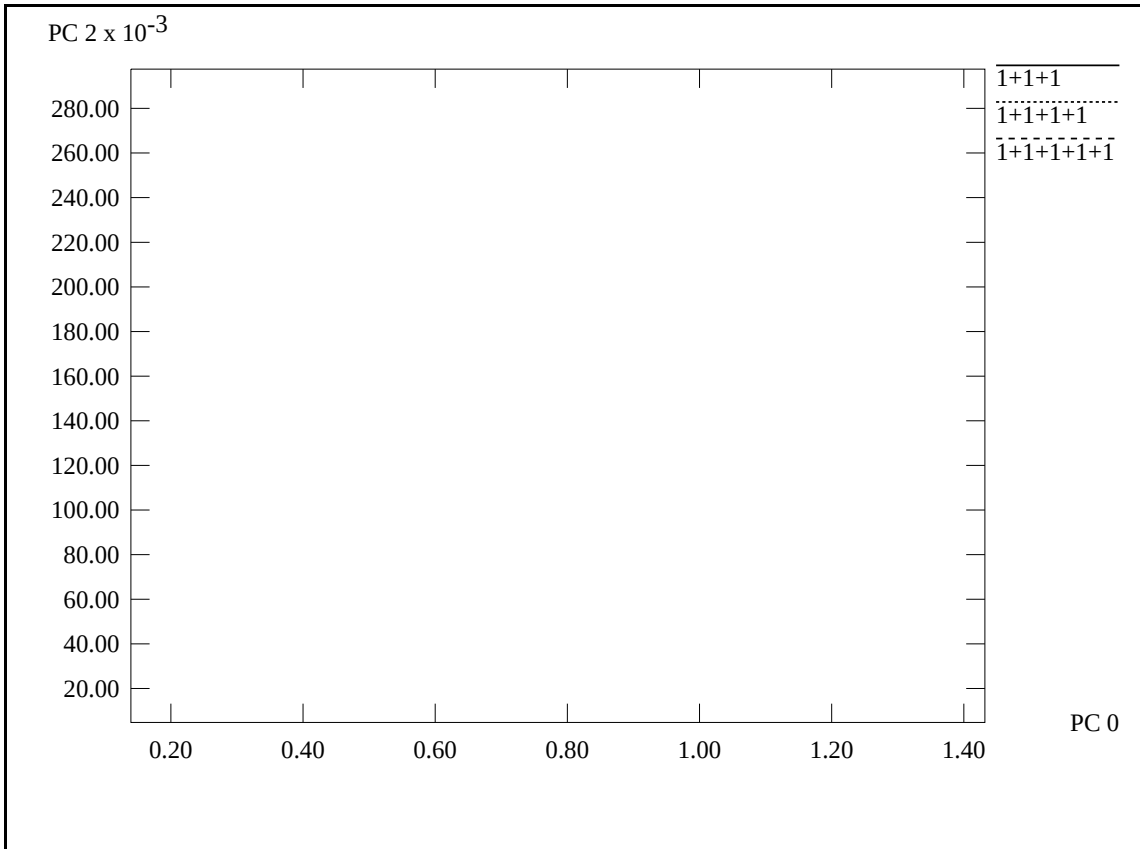
Pausible combinations

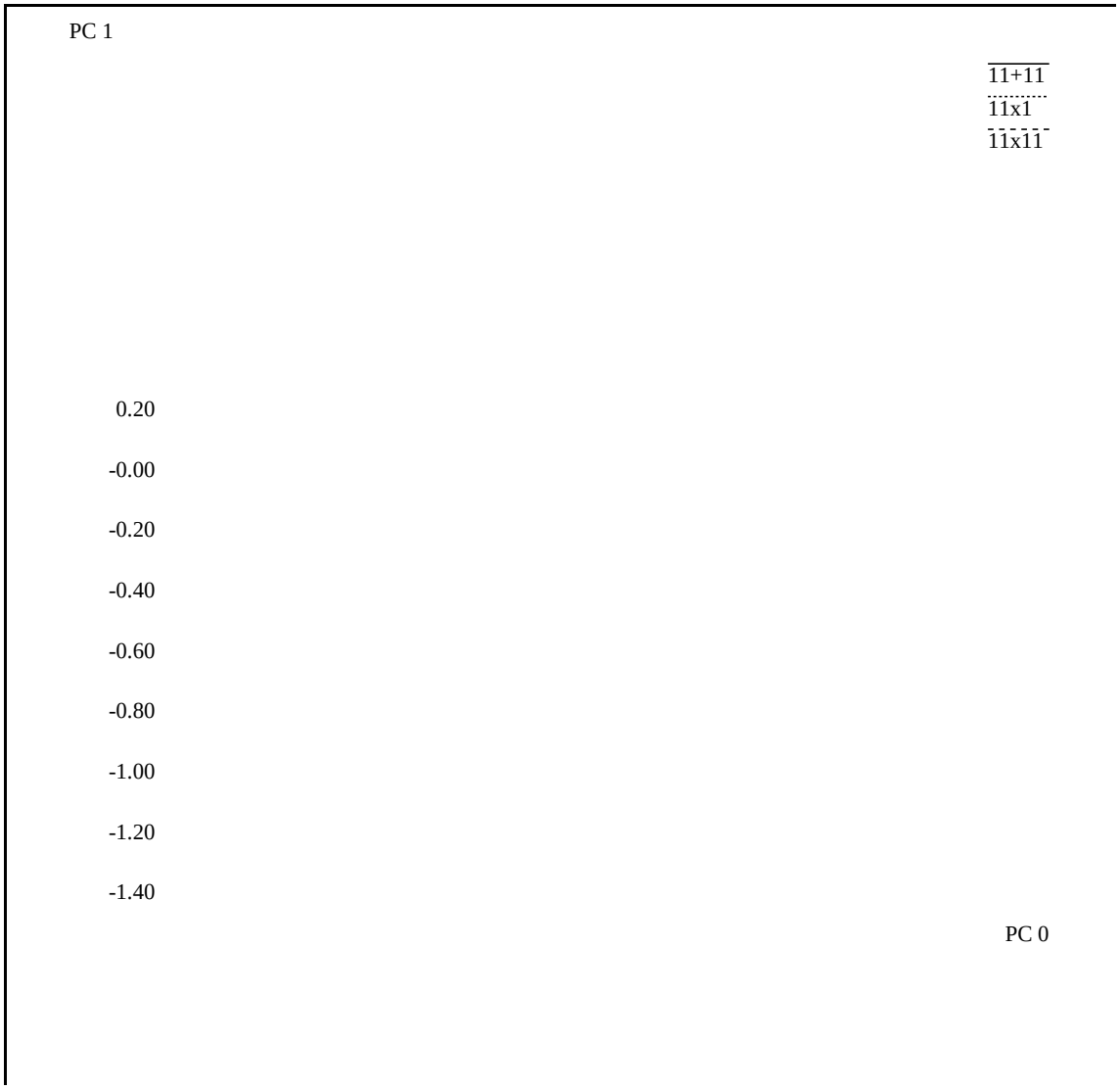
C C C C C C C C ,

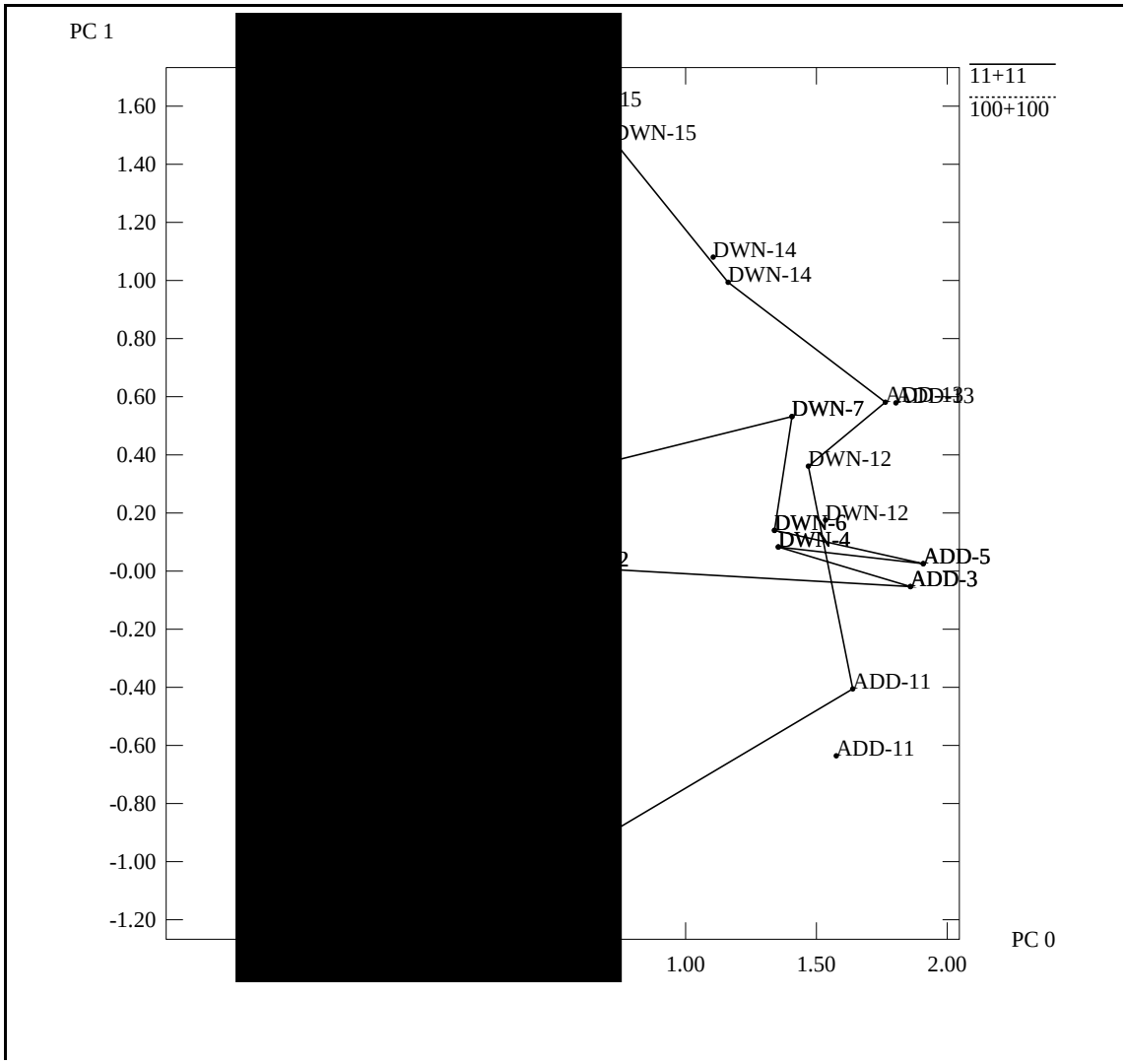
C ,– , C –
C , C, ,C – —
C , C, C ,— Б Б Б .
C , C, , .


0
undef
16
ZAC
15
SAD
14
ADD
13
DWN
12
ADD
11
DWN
10
ADD




PC 2 x 10-3
280.00
260.00
240.00
220.00
200.00
180.00
160.00
140.00
120.00
100.00
80.00
60.00
40.00
20.00
0.20
0.40
0.60
0.80
1.00
1.20
1.40
PC 0
1+1+1
1+1+1+1
1+1+1+1+1


PC 1
11+11
11x1
11x11
0.20
-0.00
-0.20
-0.40
-0.60
-0.80
-1.00
-1.20
-1.40
PC 0


PC 1
PC 0
11+11
100+100
1.60
1.40
1.20
1.00
0.80
0.60
0.40
0.20
-0.00
-0.20
-0.40
-0.60
-0.80
-1.00
-1.20
1.00
1.50
2.00
DWN-14
DWN-14
DWN-7
DWN-12
DWN-12
DWN-6
DWN-4
ADD-5
ADD-3
ADD-11
ADD-11

C C C

C C C

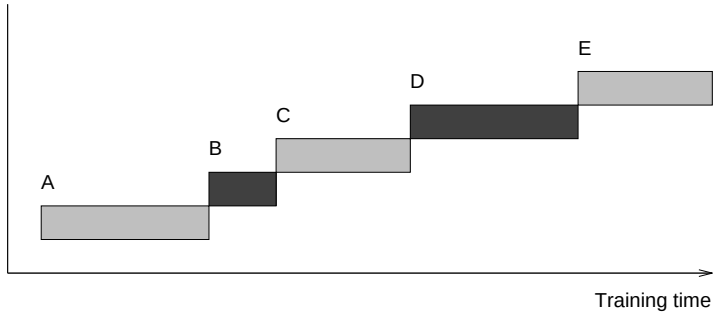


Figure C C C C C ,
C C C

C C C

5.7 Summary

5

Su ary

C

C C C C C

C C C C C

C

6.1 Memory for arithmetic facts

C C C C

Bibliography

C C C

. C C C C , C
C ,
C

. C C ,
C C *Instructiona, Science* . .

. C C C
C . C C C
C C C C C *Brain Deve,op ent and Cognition A Reader* C

C . *Co puter Mode,s of Mind*

C C . C C C C
Cognitive Science .

C . C C C C
C *Cognitive Science* . 5

C . C C
C *Inte,,igent Tutoring Syste s* C C

5 *Diagnostic Studies in Arith etic* C C
C

. C C C
C C C *Cognitive Processes in Mathe atics*
– C C

. C C
C *Canadian Journa, of Psycho,ogy* 55

. C C C C
C C *Connection Science* C

C . . C C C
C C *Neura, Connections Menta, Co putation*

. . CC C C
. CC C C C C C

5 CC C *Mind Language* 5

. C C C
Advances in Arti cia, Inte,,igence C C

. . *Mind Language* —

. . *Microcognition Phi,osophy Cognitive Science and Para,,e, Distributed Processing*

C C C C
C , C ,

C 5 C C C
Proceedings of the Eight Annua, Conference of the Cognitive Science Society

C C C *Infor a tion Thought and Content*

C C , C C

C • . C C C C •
Advances in Neura, Infor ation Processing Syste s C C
C

C • . C C C
Arti cia, Inte,,igence –5

. *Co putation and Cognition Towards a Foundation for Cognitive Science*

. C C C C C C C C C
C C *Psycho,ogica, Review* –

• C . *The Psycho,ogy of Mathe atics for Instruction*
C

• C C C C
C C ., C C
C CC C C

C . C C C •
C • C *Proceedings of the Connectionist Mode,s Su er Schoo,* 5 5 C C

Machine Learning

Machine Learning Proceedings of the Eighth International Workshop

Proceedings of the Thirteenth Annual Conference of the Cognitive Science Society

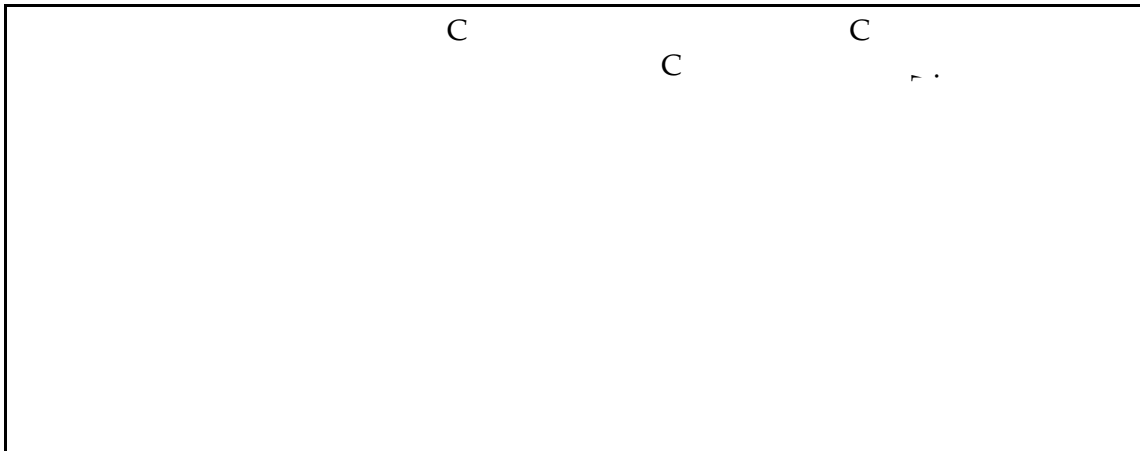
Cognitive Processes in Mathematics

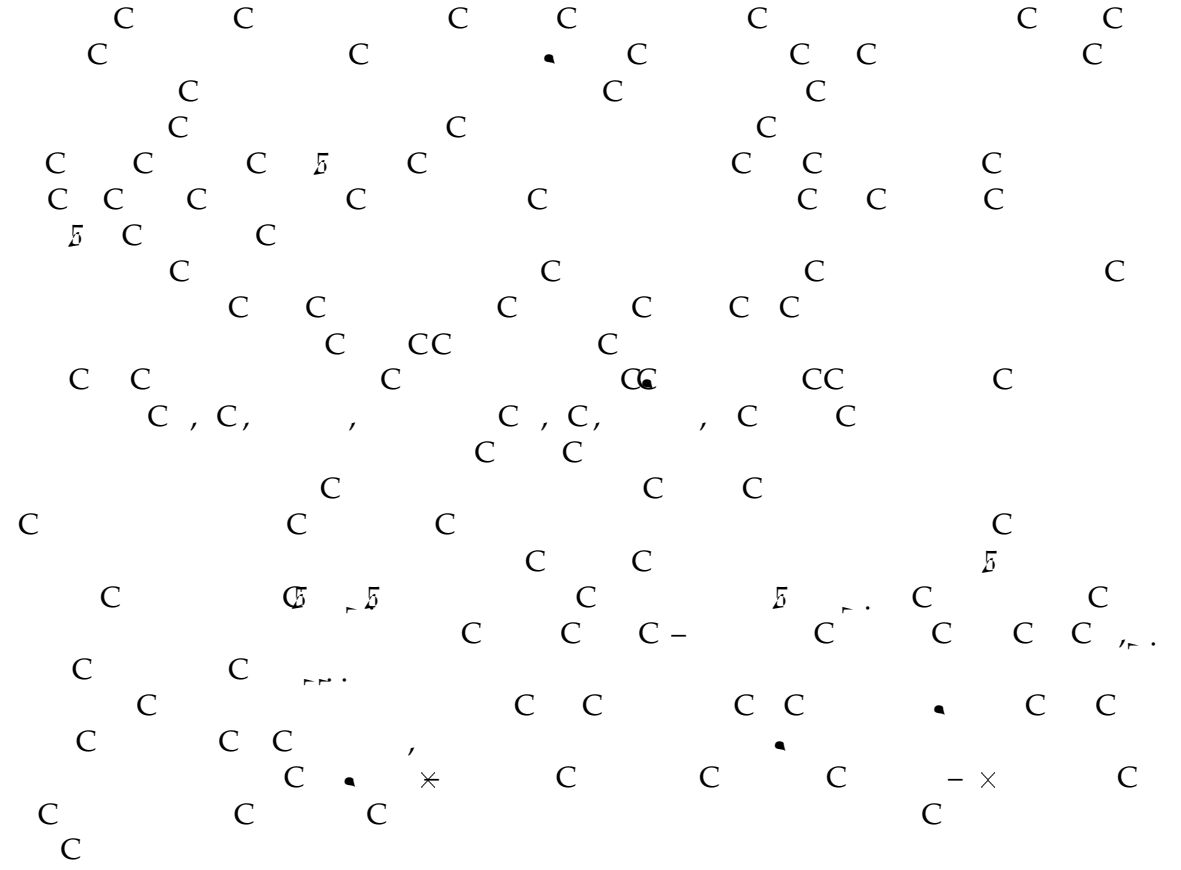
Journal of Experimental Psychology General

Origins of Cognitive Skills The Eighteenth Annual Carnegie Symposium of Cognition

Intelligent Tutoring Systems

C C C


C
C
C



Addition bugs

Added-imaginary-column.

$\cancel{5}$.

$$\begin{array}{r} 4\ 6 \\ +\quad 3 \\ \hline 7\ 9 \end{array}$$

C

Carries-one-to-100s. C C C C C

$$\begin{array}{r} 5\ 0\ 5 \\ +\quad 7\ 4 \\ \hline 6_1\ 7\ 9 \end{array}$$

C – $\cancel{5}$

Carries-one-to-10s. C C C

$$\begin{array}{r} 4\ 6 \\ +\quad 3 \\ \hline 5_1\ 9 \end{array}$$

C

Carries-ten. C –

$$\begin{array}{r} 2\ 5 \\ +\ 1\ 7 \\ \hline 1\ 3\ 2 \end{array}$$

Carries-two. C C

$$\begin{array}{r} 2\ 7\ 1 \\ +\ 4\ 1\ 2 \\ \hline 8_2\ 0\mathrm{r}870406(\mathrm{e})5 \end{array}$$

Column-skipped. C C C

$$\begin{array}{r} 3\ 7\ 5 \\ +\ 2\ 1\ 2 \\ \hline 5\quad 7 \end{array}$$

Does-not-raise-carry. C C C C C C

$$\begin{array}{r} 7\ 8 \\ +\ 7\ 1 \\ \hline {\scriptstyle 1}\ 4\ 9 \end{array}$$

Does-not-record-100s. C C C C C

$$\begin{array}{r} 5\ 0\ 5 \\ +\ \ 7\ 4 \\ \hline 0\ 7\ 9 \end{array}$$

$$\begin{array}{r} 4\ 7\ 6 \\ +\ \ 1\ 7 \\ \hline 9_{1}\ 3 \end{array}$$

C

Does-not-rename-copy-100s. C C C C C C C C C C

$$\begin{array}{r} 2\ 0\ 5 \\ +\ \ 8\ 6 \\ \hline 2\ 1\ 1 \end{array}$$

Does-not-rename-quits-100s. C C C C C C C

$$\begin{array}{r} 2\ 0\ 5 \\ +\ \ 8\ 6 \\ \hline 8\ 1\ 1 \end{array}$$

Does-not-rename-sum. C C C C

$$\begin{array}{r} 4\ 8 \\ +\ \ 3 \\ \hline 4\ 1\ 1 \end{array}$$

$$\begin{array}{r} 2\ 8 \\ \times\ 1\ 7 \\ \hline {\scriptstyle 1_1}\ 9_5\ 6 \\ +\ 2\ 8\ 0 \\ \hline 3\ 1\ 7\ 6 \end{array}$$

C

Ignores-10s-column. C C

$$\begin{array}{r} 4\ 8 \\ +\ \ 3 \\ \hline 1\ 1 \end{array}$$

Ignores-first-column. C C C C

$$\begin{array}{r} 3\ 2\ 5 \\ +\ 2\ 7\ 1 \\ \hline 5\ 9 \end{array}$$

Left-alignment. C C

$$\begin{array}{r} 5\ 4 \\ +\ 3 \\ \hline 8\ 4 \end{array}$$

C			
•			
	C , C, ,C		
	C , C,	Б–	

C			
✶	–		
,C ,C , C	Б		

Multiplication Bugs

0×N=0-carry-N. C C C

$$\begin{array}{r} 2\ 0 \\ \times\quad 3 \\ \hline 9_3\ 0 \end{array}$$

C 5

Adds-carry-and-multiplicand. C

C 5 × C

$$\begin{array}{r} 5\ 3\ 6 \\ \times\quad 8 \\ \hline 5\ 7_4\ 8 \end{array}$$

Adds-carry-and-multiplier. C

C × − 5 ×

$$\begin{array}{r} 8\ 0\ 5 \\ \times\quad 4 \\ \hline 3\ 2\ 6_2\ 0 \end{array}$$

C

Adds-carry-and-multiplier-when-zero. C

CC ×

× −

$$\begin{array}{r} 5\ 0\ 7 \\ \times\quad 2 \\ \hline 1\ 0\ 3_1\ 4 \end{array}$$

C −

Adds-carry-to-multiplicands. C C

5 ×

$$\begin{array}{r} 5\ 3\ 6 \\ \times\quad 8 \\ \hline 9\ 7_4\ 8 \end{array}$$

Adds-carry-to-product. C C C C
× 5

```
   5 2
 × 1 3
 -----
   6 6
+5 2 0
 -----
   5 8 6
```

C

Adds-instead-of-multiplying. C C C C

```
   7 2 5
 ×     3
 -------
   7 2 8
```

C 5

Adds-multiplicand-to-answer. C C
×5

```
   7 6
 ×   3
 -----
   8 8
     1
```

Adds-using-multiplication-pattern. C C

```
   3 2 0
 ×     4
 -------
   7 6 4
```

C

Always-carries.

```
 2 4 2 9
       2
 -------
 5 9 5 8
     1
```

–

Always-carries-one. C C C C C

```
   5 1 4
 ×     7
 -------
 3 5 8 8
     2
```

Answer-on-one-row. C C C C

```
     2 3
   × 4 8
 ---------
 9 3 8 4
 1 1 2
```

C 5

Answers-left-to-right. C ×. CC

```
   7 1 2
```

Carries-wrong-digit. C C C C C C C

$$\begin{array}{r} 7\ 2\ 4 \\ \times\quad\ 6 \\ \hline 4\ 8_{6}\ 1_{4}\ 2 \end{array}$$

Carries-wrong-number. C C C C C C C C C , C , C C , ,C

Carry-added-to-multiplicand. C C ,

$$\begin{array}{r} 3\ 2\ 7 \\ \times\quad\ 6 \\ \hline 3\ 6_{3}\ 6_{4}\ 2 \end{array}$$

Carry-added-to-tens. C C C

$$\begin{array}{r} 2\ 6 \\ \times\ 1\ 4 \\ \hline 2\ 8_{2}\ 4 \\ +\ 2\ 6\ 0 \\ \hline 5_{1}\ 4\ 4 \end{array}$$

Carry-not-raised. C C C C

$$\begin{array}{r} 4\ 2 \\ \times\ 4\ 1 \\ \hline 4\ 2 \\ +_{1}6\ 8\ 0 \\ \hline 7_{1}\ 2\ 2 \end{array}$$

Carry-once-always-carry. C

$$\begin{array}{r} 1\ 1\ 2 \\ \times\quad\ 7 \\ \hline 8_{1}\ 8_{1}\ 4 \end{array}$$

Copies-after-first-column. C C C C C C , –

Copies-multiplicand. C C C C C
C

$$\begin{array}{r} 200 \\ \times \quad 4 \\ \hline 200 \end{array}$$

Does-not-carry-in-partial-product. C C
C

$$\begin{array}{r} 9\ 2\ 7 \\ \times\quad 7\ 3 \\ \hline 2\ 7\ 8\ 1 \\ 6\ 4\ 8\ 9\ 0 \\ \hline 6\ 6\ 5\ 7\ 1 \end{array}$$

Does-not-carry-to-10s. C C C C

C

$$\begin{array}{r} 2\ 1\ 6 \\ \times\quad\ 6 \\ \hline 1\ 2\ 6_3\ 6 \end{array}$$

Incorrect-number-of-annex-zeros. C C C CC C
C

$$\begin{array}{r} 4\ 5\ 6 \\ \times\ 2\ 5\ 1 \\ \hline 4\ 5\ 6 \\ 2\ 2\ 8\ 0\ 0 \\ 9\ 1\ 2\ 0\ 0 \end{array}$$

C 5 . .

Last-digits-multiplied.

× × −

×

$$\begin{array}{r} 5\ 0\ 7 \\ \times\ \ 3\ 2 \\ \hline 1\ 5\ 1_1\ 4 \end{array}$$

C

Last-multiplication-skipped. C C C

$$\begin{array}{r} 3\ 2 \\ \times\ 4\ 1 \\ \hline 3\ 2 \\ +\ 8\ 0 \\ \hline 1\ 1\ 2 \end{array}$$

C

Multiplied-product-by-carry. C

Multiplies-last-multiplicand-and-writes-10. C C C C

×5 C

C

$$\begin{array}{r} 3\ 0 \\ \times\quad 6 \\ \hline 1\ 0\ 1\ 8 \end{array}$$

C –

Multiplies-multiplicands. C C ,

× ×

$$\begin{array}{r} 2\ 4 \\ \times\ 3\ 1 \\ \hline 8\ 4 \end{array}$$

C

Multiplies-partial-product. C C

, , C ,

$$\begin{array}{r} 3\ 2 \\ \times\ 2\ 1 \\ \hline 3\ 2 \\ 6\ 4\ 0 \\ \hline 7_1\ 2\ 0 \end{array}$$

C 5

Multiplies-using-addition-pattern. C

$$\begin{array}{r} 5\ 2\ 4 \\ \times\ 7\ 3\ 1 \\ \hline 3\ 5\ 6\ 4 \end{array}$$

C

× − × C C C

C C

$$\begin{array}{r} 1\ 4\ 4 \\ 2\ 5 \\ \hline 3_1\ 0_2\ 0 \end{array}$$

$$\begin{array}{r} 5\ 1\ 2 \\ \times\quad 2\ 5 \\ \hline 5\ 1\ 2 \end{array}$$

Skips-zero-multiplicand. C C C

C

C × . × –

$$\begin{array}{r} 8\ 0\ 9 \\ \times\quad 5\ 2 \\ \hline 4\ 0\ 1\ 8 \end{array}$$

Spurious-zero-in-100s. C C C C C

C –

$$\begin{array}{r} 9\ 0\ 5 \\ \times\quad 4\ 6 \\ \hline 5\ 4\ 0\ 3\ 0 \\ 3\ 6\ 0\ 2\ 0 \end{array}$$

Subtracts-partial-product. C

C C

C –

$$\begin{array}{r} 5\ 3 \\ \times\ 7\ 4 \\ \hline 2\ 1\ 2 \\ 3\ 7\ 1\ 0 \\ \hline 3\ 5\ 0\ 2 \end{array}$$

Too-many-annex-zeros.